

## INTRODUCTION

### UNIT 1 INTRODUCTION TO OPERATIONS RESEARCH

- 1.0 Introduction
- 1.1 Unit Objectives
- 1.2 Operations Research: Meaning, Nature and Origin
  - 1.2.1 Nature of Operations Research
- 1.3 Development of Operational Research
- 1.4 Operations Research in India
- 1.5 Operations Research as a Tool in Decision-Making
- 1.6 Operations Research and Management
  - 1.6.1 Significance of Operations Research
  - 1.6.2 Operations Research and Modern Business Management
- 1.7 Features and Methodology of Operations Research and Phases of Operations Research Study
  - 1.7.1 Methodology of Operations Research
- 1.8 Models in Operations Research and Methods of Deriving the Solution
  - 1.8.1 Advantages of a Model
  - 1.8.2 Classification of Models
- 1.9 Limitations of Operations Research
- 1.10 Summary
- 1.11 Key Terms
- 1.12 Answers to 'Check Your Progress'
- 1.13 Questions and Exercises
- 1.14 Further Reading

### UNIT 2 LINEAR PROGRAMMING, DUALITY AND SENSITIVITY ANALYSIS

33-68

- 2.0 Introduction
- 2.1 Unit Objectives
- 2.2 Linear Programming: Meaning and Uses
  - 2.2.1 Meaning of Linear Programming
  - 2.2.2 Fields Where Linear Programming can be Used
- 2.3 Concepts, Notations and General Form of Linear Programming Model
  - 2.3.1 Basic Concepts and Notations
  - 2.3.2 General Form of the Linear Programming Model
- 2.4 Applications and Limitations of Linear Programming Problems
- 2.5 Formulation of Linear Programming Problem
  - 2.5.1 Graphic Solution
  - 2.5.2 General Formulation of Linear Programming Problem
  - 2.5.3 Matrix Form of Linear Programming Problem
- 2.6 Solution of Linear Programming Problem: Graphical Solution and Simplex Method
  - 2.6.1 Graphical Solution
  - 2.6.2 Some Important Definitions
  - 2.6.3 Canonical or Standard Forms of LPP
  - 2.6.4 Simplex Method
  - 2.6.5 *M* Method
- 2.7 Duality
- 2.8 Sensitivity Analysis
- 2.9 Summary

- 2.10 Key Terms
- 2.11 Answers to 'Check Your Progress'
- 2.12 Questions and Exercises
- 2.13 Further Reading

### UNIT 3 TRANSPORTATION PROBLEM

71-116

- 3.0 Introduction
- 3.1 Unit Objectives
- 3.2 Basics of Transportation Problem
  - 3.2.1 Transportation Algorithm
- 3.3 Applications of Transportation Problem
  - 3.3.1 The Travelling Salesman Problem
  - 3.3.2 Mathematical Formulation
- 3.4 Solutions of Transportation Problem
  - 3.4.1 North West Corner Rule
  - 3.4.2 Least Cost or Matrix Minima Method
  - 3.4.3 Vogel's Approximation Method (VAM)
- 3.5 MODI Method
- 3.6 Special Cases
  - 3.6.1 Maximization Case in Transportation Problem
  - 3.6.2 Transshipment Problem
  - 3.6.3 Transshipment Model
- 3.7 Summary
- 3.8 Key Terms
- 3.9 Answers to 'Check Your Progress'
- 3.10 Questions and Exercises
- 3.11 Further Reading

### UNIT 4 ASSIGNMENT PROBLEM

119-149

- 4.0 Introduction
- 4.1 Unit Objectives
- 4.2 Basics of Assignment Problem
  - 4.2.1 Mathematical Formulation of the Assignment Problem
- 4.3 Applications and Solutions by Hungarian Method
  - 4.3.1 Hungarian Method Procedure
  - 4.3.2 Maximization in Assignment Problem
- 4.4 Job Sequence Problem
  - 4.4.1 Unbalanced Assignment Problem
- 4.5 Variations of the Assignment Problem
- 4.6 Summary
- 4.7 Key Terms
- 4.8 Answers to 'Check Your Progress'
- 4.9 Questions and Exercises
- 4.10 Further Reading

### UNIT 5 NETWORK ANALYSIS

151-204

- 5.0 Introduction
- 5.1 Unit Objectives
- 5.2 Introduction to Network Concept
  - 5.2.1 Development of Network Analysis - CPM and PERT
- 5.3 Network Analysis and Rules of Network Construction
  - 5.3.1 Rules of Network Construction
  - 5.3.2 Time Analysis
  - 5.3.3 Network Diagram

### 5.4 Critical Path Method (CPM)

- 5.4.1 Computations for Critical Path
- 5.4.2 Applications of CPM Analysis
- 5.5 Programme Evaluation and Review Technique (PERT)
  - 5.5.1 PERT Procedure
- 5.6 Comparison and Limitations of PERT and CPM
- 5.7 Basic Concepts of Crashing
- 5.8 Resource Leveling and Resource Smoothing
- 5.9 Summary
- 5.10 Key Terms
- 5.11 Answers to 'Check Your Progress'
- 5.12 Questions and Exercises
- 5.13 Further Reading

### UNIT 6 INVENTORY, QUEUING MODELS AND SIMULATION

- 6.0 Introduction
- 6.1 Unit Objectives
- 6.2 Queuing Models
  - 6.2.1 Kendall's Notation for Representing Queuing Models
  - 6.2.2 Classification of Queuing Models
- 6.3 Simulation
  - 6.3.1 Simulation of a Queuing System Using Event List
  - 6.3.2 Multiplicative Congruential Algorithm
  - 6.3.3 Inverse Transformation Method
- 6.4 Monte-Carlo Simulation
- 6.5 Inventory Problems
  - 6.5.1 EOQ with Price Breaks
- 6.6 Summary
- 6.7 Key Terms
- 6.8 Answers to 'Check Your Progress'
- 6.9 Questions and Exercises
- 6.10 Further Reading

# INTRODUCTION

Operations Research, or OR, originated in the context of military operations, but today is widely accepted as a powerful tool for planning and decision-making, especially in business and industry. The OR approach has provided a new tool for managing conventional management problems. In fact, operational research techniques constitute a scientific methodology of analysing the problems of the business world and provide an improved basis for taking management decisions. The practice of OR helps in tackling intricate and complex problems such as that of resource allocation, product mix, inventory management, sequencing and scheduling, replacement and a host of similar problems of modern business and industry.

Operations research is an interdisciplinary branch of applied mathematics and formal science that uses mathematical methods, such as mathematical modelling, statistics and algorithms, to arrive at optimal or near optimal solutions to complex problems. Basically, it is concerned with optimizing the maxima (profit, assembly line performance, bandwidth, etc) or minima (loss, risk, etc.) of some objective function. It also helps management achieve its goals using scientific methods. The field of operations research is closely related to industrial engineering, and industrial engineers consider operations research techniques as their major toolset. Some of the primary tools used by operations researchers are statistics, optimization, probability theory, queuing theory, game theory, graph theory, decision analysis and simulation. Because of the computational nature of these fields, OR is linked to computer science and OR professionals use specific custom-written software for computation of data and decision-making.

The uniqueness of OR prompted industries to use its formal tools such as operations analysis, system analysis, management science, decision science, etc. Commercial industries such as airlines, automobiles, communications, electronics, transportation, chemicals and mining use OR techniques to optimally utilize their limited resources and thereby maximize profits. Hence, OR is the application of the methods of science to complex problems arising in the direction and management of large systems of men, machines, materials and money in industry, business, government and defence. The distinctive approach is to develop a scientific model of the system, incorporating measurement of factors such as chance and risk, with which to predict and compare the outcomes of alternative decision strategies and controls.

Operations research provides top-level administrators a quantitative basis for taking decisions which will help organizations to carry out their functions such as planning, controlling and organizing, effectively. Decision-making is the key responsibility of managers and OR provides a scientific approach to them for solving problems. Decisions in an organization should be such that they can compete in the market. We can say that the OR and decision-making are interlinked. There are intangible factors also, such as human behaviour, which OR has to take into account when calculating for a solution.

This book, *Operations Research*, provides a basic understanding of the subject and helps to grasp its fundamentals. In a nutshell, it explains various aspects of operations research, models in operations research, applications and limitations of operations research, linear programming problem, duality in linear programming problem, sensitivity analysis, transportation problem, assignment problem, network

analysis (CPM and PERT), simulation and queueing system, and basic ideas of Monte-Carlo simulation. The concepts are analysed in a logical format, beginning with an overview that helps the reader to easily understand the concept, followed by explanations and solved examples. This book has been written in the SIM format or the self-instructional mode wherein each unit begins with an 'Introduction' to the topic followed by an outline of the 'Unit Objectives'. The detailed content is then presented in a simple and an organized manner, interspersed with 'Check Your Progress' questions to test the understanding of the students. A 'Summary' along with a list of 'Key Terms' and a set of 'Questions and Exercises' is also provided at the end of each unit for effective recapitulation. Additional explanations and examples have been provided to clarify the areas where students often have difficulty.

---

## MODULE I

---

# UNIT 1 INTRODUCTION TO OPERATIONS RESEARCH

## Structure

- 1.0 Introduction
- 1.1 Unit Objectives
- 1.2 Operations Research: Meaning, Nature and Origin
  - 1.2.1 Nature of Operations Research
- 1.3 Development of Operational Research
- 1.4 Operations Research in India
- 1.5 Operations Research as a Tool in Decision-Making
- 1.6 Operations Research and Management
  - 1.6.1 Significance of Operations Research
  - 1.6.2 Operations Research and Modern Business Management
- 1.7 Features and Methodology of Operations Research and Phases of Operations Research Study
  - 1.7.1 Methodology of Operations Research
- 1.8 Models in Operations Research and Methods of Deriving the Solution
  - 1.8.1 Advantages of a Model
  - 1.8.2 Classification of Models
- 1.9 Limitations of Operations Research
- 1.10 Summary
- 1.11 Key Terms
- 1.12 Answers to 'Check Your Progress'
- 1.13 Questions and Exercises
- 1.14 Further Reading

## 1.0 INTRODUCTION

In this unit, you will learn about Operations Research (OR), which is a very powerful tool for decision-making. The term Operations Research was coined by J.F. McCloskey and F.N. Trefethen in 1940 in Bowdsey in the United Kingdom. This innovative science was discovered during World War II for a specific military situation, when military management sought decisions based on the optimal consumption of limited military resources with the help of an organized and systematized scientific approach. This was termed as operations research or operational research. Thus, OR was known as an ability to win a war without really going into a battlefield or fighting it. It is a new field of scientific and managerial application.

You will learn the scope of OR and its applications. The different phases of applications include—implementing and formulating the problem, constructing a mathematical model, deriving the result from the model, testing and updating the model and controlling the final output or solution. You will learn about the requirements of a good model which must be capable of working on a new formulation without making any changes in its frame with minimum assumptions and with minimum variables and must not take extraordinary time to solve the problem.

You will understand how the various models of OR work and the techniques adopted. These techniques are: linear programming, waiting line or queuing theory,

## NOTES

inventory control/planning, game theory, decision theory, network analysis, simulation, integrated production models, non-linear programming, dynamic programming, heuristic programming, integer programming, algorithmic programming, quadratic programming, parametric programming, probabilistic programming, search theory and theory of replacement. All these techniques involve higher mathematics. In real practice these techniques are used in combination to form more sophisticated and advanced programming models.

### 1.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand the definition and scope of operations research (OR)
- Define development of OR
- Analyse the nature of OR
- Know about OR in India
- Describe OR as a tool in decision-making
- Explain the relation between OR and management
- Know about the phases and applications of OR
- Understand the requirements of a good model
- Understand the working of various models and techniques of OR
- Derive solutions using OR

### 1.2 OPERATIONS RESEARCH: MEANING, NATURE AND ORIGIN

The term, operations research was first coined in 1940 by J.F. McCloskey and F.N. Trefethen in a small town Bowdsey in the United Kingdom. This new science came into existence in a military context. During World War II, military management called on scientists from various disciplines and organized them into teams to assist in solving strategic and tactical problems, relating to air and land defence. Their mission was to formulate specific proposals and plans for aiding the Military commands to arrive at decisions on optimal utilization of scarce military resources and attempts to implement these decisions effectively. This new approach to the systematic and scientific study of the operations of the system was called Operations Research (OR) or Operational Research. Hence, OR can be associated with 'an art of winning the war without actually fighting it'.

#### Definitions

Operations Research (OR) has been defined so far in various ways and it is perhaps still too young to be defined in some authoritative way. It is not possible to give uniformly acceptable definitions of OR. A few opinions about the definition of OR are given below. These have been changed according to the development of the subject.

OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.

- Morse and Kimball (1946)

OR is the scientific method of providing executive with an analytical and objective basis for decisions.

P.M.S. Blackett (1948)

OR is a systematic method-oriented study of the basic structures, characteristics, functions and relationships of an organization to provide the executive with a sound, scientific and quantitative basis for decision-making.

E.L. Arnoff and M.J. Netzorg

OR is a scientific approach to problem solving for executive management.

H.M. Wagner

OR is an aid for the executive in making his decisions by providing him with the quantitative information based on the scientific method of analysis.

C.Kittee

OR is the scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.

H.A. Taha

The various definitions given here bring out the following essential characteristics of operations research:

- (i) System orientation
- (ii) Use of interdisciplinary terms
- (iii) Application of scientific methods
- (iv) Uncovering new problems
- (v) Quantitative solutions
- (vi) Human factors

#### Scope of Operations Research

There is a great scope for economists, statisticians, administrators and the technicians working as a team to solve problems of defence by using the OR approach. Besides this, OR is useful in various other important fields like:

- (i) Agriculture
- (ii) Finance
- (iii) Industry
- (iv) Marketing
- (v) Personnel management
- (vi) Production management
- (vii) Research and development

#### Phases of Operations Research

The procedure to be followed in the study of OR generally involves the following major phases:

- (i) Formulating the problem
- (ii) Constructing a mathematical model
- (iii) Deriving the solution from the model
- (iv) Testing the model and its solution (updating the model)
- (v) Controlling the solution
- (vi) Implementation

### 1.2.1 Nature of Operations Research

Looking to the basic features of the definitions concerning OR, we can state that, 'Operational Research can be considered as the application of scientific method by interdisciplinary teams to problems involving the control of organized (man-machine) systems to provide solutions, which best serve the purposes of the organization as a whole'.

Different characteristics constituting the nature of OR can be summed up as follows:

1. **Interdisciplinary team approach:** The Operations Research has the characteristics that it is done by a team of scientists drawn from various disciplines such as mathematics, statistics, economics, engineering, physics, etc. It is essentially an interdisciplinary team approach. Each member of the OR team is benefited from the viewpoint of the other so that a workable solution obtained through such collaborative study has a greater chance of acceptance by management.
2. **Systems approach:** Operations Research emphasizes on the overall approach to the system. This characteristic of OR is often referred as system orientation. The orientation is based on the observation that in the organized systems the behaviour of any part ultimately has some effect on every other part. But all these effects are not significant and even not capable of detection. Therefore, the essence of system orientation lies in the systematic search for significant interactions in evaluating actions of any part of the organization. In OR an attempt is made to take account of all the significant effects and to evaluate them as a whole. OR thus considers the total system for getting the optimum decisions.
3. **Helpful in improving the quality of solution:** Operations Research cannot give perfect answers or solutions to the problems. It merely gives bad answers to the problems which otherwise have worst answers. Thus, OR simply helps in improving the quality of the solution but does not result into a perfect solution.
4. **Scientific method:** Operations Research involves scientific and systematic attack of complex problems to arrive at the optimum solution. In other words, Operations Research or OR uses techniques of scientific research. Thus OR comprehends both aspects, i.e., it includes both scientific research on the phenomena of operating systems and the associated engineering activities aimed at applying the results of research.
5. **Goal oriented optimum solution:** Operations Research tries to optimize a well-defined function subject to given constraints and as such is concerned with the optimization theory.

6. **Use of models:** Operations Research uses models built by quantitative measurement of the variables concerning a given problem and also derives a solution from the model using one or more of the diversified solution techniques. A solution may be extracted from a model either by conducting experiments on it or by mathematical analysis. The purpose is to help the management to determine its policy and actions scientifically.

7. **Require willing executives:** Operations Research does require the willingness on the part of the executive for experimentation to evaluate the costs and the consequences of the alternative solutions of the problem. It enables the decision-maker to be objective in choosing an alternative from among many possible alternatives.

8. **Reduces complexity:** Operations Research tries to reduce the complexity of business operations and does help the executive in correcting a troublesome function and to consider innovations which are too costly and complicated to experiment with the actual practice.

In view of the above, OR must be viewed as both a science and an art. As science, OR provides mathematical techniques and algorithms for solving appropriate decision problems. OR is an art because success in all the phases that precede and succeed the solution of a problem largely depends on the creativity and personal ability of the decision-making analysts.

### 1.3 DEVELOPMENT OF OPERATIONAL RESEARCH

The subject of Operational Research (OR) was developed in military context during World War II, pioneered by the British scientists. At that time, the military management in England appointed a study group of scientists to deal with the strategic and tactical problems related to air and land defence of the country. The main reason for conducting the study was that they were having very limited military resources. It was, therefore, necessary to decide upon the most effective way of utilizing these resources. As the name implies, Operations Research was apparently invented because the team was dealing with research on military operations. The scientists studied the various problems and on the basis of quantitative study of operations suggested certain approaches which showed remarkable success. The encouraging results obtained by the British operations research teams consisting of personnel drawn from various fields like Mathematics, Physics, Biology, Psychology and other physical sciences, quickly motivated the United States military management to start similar activities. Successful innovations of the US teams included the development of new flight patterns, planning sea mining and effective utilization of electronic equipments. Similar OR teams also started functioning in Canada and France. These OR teams were usually assigned to the executive-in-charge of operations and as such their work came to be known as 'Operational Research' in the UK and by a variety of names in the United States an Operational Analysis, Operations Evaluation, Operations Research, Systems Analysis, Systems Evaluation and Systems Research. The name 'Operational Research' or 'Operations Research' or simply OR is most widely used now a days all over the world for the systematic and scientific study of the operations of the system. Till fifties, use of OR was mainly confined to military purposes.

#### Check Your Progress

1. How was the concept of operations research started?
2. State one definition of OR.
3. Mention the essential characteristics of operations research.
4. Apart from defence, mention other areas where OR is applied.
5. What are major phases in the application of OR?

## NOTES

After the end of the second world war, the success of military teams attracted the attention of industrial managers who were seeking solutions to their complex managerial problems. At the end of the war, expenditures on defence research were reduced in the UK and this led to the release of many operations research workers from the military at a time when industrial managers were confronted with the need to reconstruct most of Britain's manufacturing industries and plants that had been damaged in war. Executives in such industries sought assistance from the said operations research workers. But in the USA most of the war experienced operations research workers remained in military service as the defence research was increased and consequently operations research was expanded at the end of the war. It was only in the early 1950s, industry in the USA began to absorb the operations research worker under the pressure for increased demands for greater productivity originated because of the outbreak of the Korean conflict and because of technological developments in industry. Thus, OR began to develop in industrial field in the United States since the year 1950. The Operations Research Society of America was formed in 1953 and in 1957 the International Federation of Operational Research Societies was established. Various journals relating to OR began to appear in different countries in the years that followed the mid-fifties. Courses and curricula in OR in different universities and other academic institutions began to proliferate in the United States. Other countries rapidly followed this and after the late fifties Operations Research was applied for solving business and industrial problems. Introduction of Electronic Data Processing (EDP) methods further enlarged the scope for application of OR techniques. With the help of a digitized computer many complex problems can be studied on day-to-day basis. As a result, many industrial concerns are adopting OR as an integrated decision-making tool for their routine decision procedures.

### 1.4 OPERATIONS RESEARCH IN INDIA

Today, the impact of OR in Indian business and industry can be felt in many areas. A large number of management consulting firms are recently engaged in OR activities. Apart from military and business applications, OR activities include transportation systems, libraries, hospitals, city planning, financial institutions, etc. With increasing use of computers the operations research techniques have started playing a noticeable role in our country as well. The major Indian industries such as Delhi Cloth Mills, Indian Railways, Indian Airlines, Defence Organizations, Hindustan Liver, Tata Iron and Steel Company, Fertilizer Corporation of India and similar industries make use of operations research techniques for solving problems and making decisions.

Historically, Operations Research started developing in India after independence specially with the setting up of an Operation Research Unit at the Regional Research Laboratory at Hyderabad in 1949. Operations Research activities gained further impetus with the establishment of an Operations Research Unit in 1953 in the Indian Statistical Institute (ISI), Calcutta for applying Operations Research techniques in national planning and survey. Operational Research Society of India was formed in 1957 which joined International Federation of Operational Research Societies in 1960 by becoming its member. The said society helped the cause of the development of Operations Research activities in India in several ways and started publishing a journal

of Operations Research entitled 'OPSEARCH' from 1963. Besides, the Indian Institute of Industrial Engineers has also promoted Operations Research in India and its journals viz., 'Industrial Engineering' and 'Management' are considered as important key journals relating to Operations Research in the country. Other important journals which deal with Operations Research in our country are the Journal of the National Productivity Council, Materials Management Journal of India and the Defence Science Journal. There are several institutions which train and produce people in the field of Operations Research to meet the need of OR practitioners in the country.

So far as the application of Operations Research in India is concerned it was Professor P.C. Mahalanobis of ISI, Calcutta who made the first important application. He formulated the Second Five Year Plan of our country with the help of OR technique to forecast the trends of demand, availability of resources and for scheduling the complex scheme necessary for developing our country's economy. It was estimated that India could become self-sufficient in food and solve her foreign exchange problems merely by reducing the wastage of food by 15%. Operational Research Commission made the use of OR techniques for planning the optimum size of the Caravelle fleet of Indian Airlines. Kirloskar company made use of assignments models for allocation of their salesmen to different areas so as to maximize their profit. Linear Programming (LP) models were also used by them to assemble various diesel engines at minimum possible cost. Various cotton textile leaders such as Binny, DCM, Calico, etc., are using linear programming techniques in cotton blending. Many other firms like Union Carbide, ICI, TELCO and Hindustan Liver etc., are making use of OR techniques for solving many of their complex business problems. State Trading Corporation of India (STCI) has also set up a Management Sciences Group with the idea of promoting and developing the use of OR techniques in solving its management decision problems. Besides, many Universities and professional academic institutions are imparting training in OR in our country. The subject of OR has been included in the courses of such institutions. But in comparison with the western world the present state of OR in our country is much behind. Operations Research activities are very much limited and confined only to the big organized industries. Most popular practical application of Operations Research has been mainly that of Linear Programming. There is relative scarcity of well-trained operational researchers. The use of Operations Research is relatively a very costly affair. In spite of several limitations, our industrialists are gradually becoming conscious of the role of Operations Research techniques and in the coming years such techniques will have an increasingly important role to play in Indian business and industry.

### 1.5 OPERATIONS RESEARCH AS A TOOL IN DECISION-MAKING

Mathematical models have been constructed for OR problems and methods for solving the models are available in many cases. Such methods are usually termed as OR techniques. Some of the important OR techniques often used by decision-makers in modern times in business and industry are as under:

1. *Linear programming*. This technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost minimization under certain

constraints. This technique is primarily concerned with the optimal allocation of limited resources for optimizing a given function. The name linear programming is given because of the fact that the model in such cases consists of linear equations indicating linear relationship between the different variables of the system. Linear programming technique solves product-mix and distribution problems of business and industry. It is a technique used to allocate scarce resources in an optimum manner in problems of scheduling, product-mix and so on. Key factors under this technique include an objective function, choice among several alternatives, limits or constraints stated in symbols and variables assumed to be linear.

2. *Waiting line or Queuing theory.* Waiting line or queuing theory deals with mathematical study of queues. The queues are formed whenever the current demand for service exceeds the current capacity to provide that service. Waiting line technique concerns itself with the random arrival of customers at a service station where the facility is limited. Providing too much of capacity will mean idle time for servers and will lead to waste of 'money'. On the other hand, if the queue becomes long there will be a cost due to waiting of units in the queue. Waiting line theory, therefore, aims at minimizing the costs of both servicing and waiting. In other words, this technique is used to analyse the feasibility of adding facilities and to assess the amount and cost of waiting time. With its help we can find the optimal capacity to be installed which will lead to a sort of an economic balance between cost of service and cost of waiting.
3. *Inventory control/Planning.* Inventory planning aims at optimizing inventory levels. Inventory may be defined as a useful idle resource which has economic value, e.g., raw materials, spare parts, finished products, etc. Inventory planning, in fact, answers the two questions, viz., how much to buy and when to buy. Under this technique the main emphasis is on minimizing costs associated with holding of inventories, procurement of inventories and the shortage of inventories.
4. *Game theory.* Game theory is used to determine the optimum strategy in a competitive situation. Simplest possible competitive situation is that of two persons playing zero-sum game, i.e., a situation in which two persons are involved and one person wins exactly what the other loses. More complex competitive situations in real life can be imagined where game theory can be used to determine the optimum strategy.
5. *Decision theory.* Decision theory concerns with making sound decisions under conditions of certainty, risk and uncertainty. As a matter of fact there are three different kinds of states under which decisions are made, viz., deterministic, stochastic and uncertainty and the decision theory explains how to select a suitable strategy to achieve some object or goal under each of these three states.
6. *Network analysis.* Network analysis involves the determination of an optimum sequence of performing certain operations concerning some jobs in order to minimize overall time and/or cost. Programme Evaluation and Review Technique (PERT), Critical Path Method (CPM) and other network techniques such as Gantt Chart come under network analysis. Key concepts under this

technique are network of events and activities, resource allocation, time and cost considerations, network paths and critical paths.

7. *Simulation.* Simulation is a technique of testing a model, which resembles a real life situation. This technique is used to imitate an operation prior to actual performance. Two methods of simulation are used—Monte Carlo method and the System simulation method. The former uses random numbers to solve problems which involve conditions of uncertainty and the mathematical formulation is impossible. In case of System simulation, there is a reproduction of the operating environment and the system allows for analysing the response from the environment to alternative management actions. This method draws samples from a real population instead of from a table of random numbers.
8. *Integrated production models.* This technique aims at minimizing cost with respect to work force, production and inventory. This technique is highly complex and is used only by big business and industrial units. This technique can be used only when sales and costs statistics for a considerable long period are available.
9. *Some other OR techniques.* In addition, there are several other techniques such as non-linear programming, dynamic programming, search theory, the theory of replacement, etc. A brief mention of some of these is as follows:
  - (i) *Non-linear programming.* A form of programming in which some or all of the variables are curvilinear. In other words, this means that either the objective function or constraints or both are not in linear form. In most of the practical situations, we encounter non-linear programming problems but for computation purpose we approximate them as linear programming problems. Even then there may remain some non-linear programming problems which may not be fully solved by presently known methods.
  - (ii) *Dynamic programming.* It refers to a systematic search for optimal solutions to problems that involve many highly complex interrelations that are, moreover, sensitive to multistage effects such as successive time phases.
  - (iii) *Heuristic programming.* It is also known as discovery method and refers to step-by-step search towards an optimum when a problem cannot be expressed in mathematical programming form. The search procedure examines successively a series of combinations that lead to stepwise improvements in the solution and the search stops when a near optimum has been found.
  - (iv) *Integer programming.* It is a special form of linear programming in which the solution is required in terms of integral numbers (i.e., whole numbers) only.
  - (v) *Algorithmic programming.* It is just the opposite of Heuristic programming. It may also be termed as similar to mathematical programming. This programming refers to a thorough and exhaustive mathematical approach to investigate all aspects of the given variables in order to obtain optimal solution.

- (vi) *Quadratic programming.* It refers to a modification of linear programming in which the objective equations appear in quadratic form, i.e., they contain squared terms.
- (vii) *Parametric programming.* It is the name given to linear programming when it is modified for the purpose of inclusion of several objective equations with varying degrees of priority. The sensitivity of the solution to these variations is then studied.
- (viii) *Probabilistic programming.* It is also known as stochastic programming and refers to linear programming that includes an evaluation of relative risks and uncertainties in various alternatives of choice for management decisions.
- (ix) *Search theory.* It concerns itself with search problems. A search problem is characterized by the need for designing a procedure to collect information on the basis of which one or more decisions are made. This theory is useful in places in which some events are known to occur but the exact location is not known. The first search model was developed during World War II to solve decision problems connected with air patrols and their search for submarines. Advertising agencies search for customers, personnel departments search for good executives are some of the examples of search theory's application in business.
- (x) *The theory of replacement.* It is concerned with the prediction of replacement costs and determination of the most economic replacement policy. There are two types of replacement models. First type of models deal in replacing equipments that deteriorate with time and the other type of models help in establishing replacement policy for those equipments which fail completely and instantaneously.

All these techniques are not simple but involve higher mathematics. The tendency today is to combine several of these techniques and form into more sophisticated and advanced programming models.

## 1.6 OPERATIONS RESEARCH AND MANAGEMENT

Operations Research plays significant role in industries and empowers management to make better decision. The following are the important applications of OR in business and management:

### 1.6.1 Significance of Operations Research

Operations Research has gained increasing importance since World War II in the technology of business and industry administration. It greatly helps in tackling the intricate and complex problems of modern business and industry. OR techniques are, in fact, examples of the use of scientific method of management. The significance of OR can be well understood under the following heads:

1. *OR provides a tool for scientific analysis.* OR provides the executives with a more precise description of the cause and effect relationship and risks underlying the business operations in measurable terms and this eliminates the conventional

intuitive and subjective basis on which managements used to formulate their decisions decades ago. In fact, OR replaces the intuitive and subjective approach of decision-making by an analytical and objective approach. The use of OR has transformed the conventional techniques of operational and investment problems in business and industry. As such OR encourages and enforces disciplined thinking about organizational problems.

2. *OR provides solutions for various business problems.* OR techniques are being used in production, procurement, marketing, finance and other allied fields. It can be used to solve problems like how best can managers and executives allocate the available resources to various products so that in a given time the profits are maximum or the cost is minimum, possibility of an industrial enterprise to arrange the time and quantity of orders of its stocks such that the overall profit with given resources is maximum, competence of business managers to determine the number of men and machines to be employed and used in such a manner that neither remains idle and at the same time the customer or the public has not to wait unduly long for service, and similar other problems. Similarly we might have a complex of industries; steel, machine tools and others, all employed in the production of one item, say, steel. At any particular time we have a number of choices of allocating resources such as money, steel and tools for producing autos, building steel factories or tool factories. What should be the policy which optimizes the total number of autos produced over a given period? OR techniques are capable of providing an answer in such a situation.

Planning decisions in business and industry are largely governed by the picture of anticipated demands. The potential long range profits of the business may vary in accordance with different possible demand patterns. The OR techniques serve to develop a scientific basis for coping with the uncertainties of future demands. Thus, in dealing with the problem of uncertainty over future sales and demands, OR can be used to generate 'a least risk' plan.

At times there may be a problem of finding an acceptable definition of long range company objectives. Management may be confronted with different viewpoints; some may stress the desirability of maximizing of net profit whereas others may focus attention primarily on the minimization of costs. OR techniques, specially that of mathematical programming such as linear programming can help resolve such dilemmas by permitting systematic evaluation of the best strategies for attaining different objectives. These techniques can also be used for estimating the worth of technical innovations as also of potential profits associated with the possible changes in rules and policies.

How much changes can there be in the data on which a planning formulation is based without undermining the soundness of the plan itself? How accurately must managements know cost coefficients, production performance figures and other factors before it can make planning decisions with confidence? Many of the basic data required for the development of long-range plans are uncertain. Such uncertainties though cannot be avoided but

through various OR techniques the management can know how critical such uncertainties are and this in itself is a great help to business planners.

3. *OR enables proper deployment of resources.* OR renders valuable help in proper deployment of resources. For example, Programme Evaluation and Review Technique (PERT) enables us to determine the earliest and the latest times for each of the events and activities and thereby helps in identification of the critical path. All this helps in the deployment of resources from one activity to another to enable the project completion on time. This technique, thus provides for determining the probability of completing an event or project itself by a specified date.
4. *OR helps in minimizing waiting and servicing costs.* The waiting line or queuing theory helps management in minimizing the total waiting and servicing costs. This technique also analyses the feasibility of adding facilities and thereby helps businesspeople take correct and profitable decisions.
5. *OR enables management to decide when to buy and how much to buy.* The main objective of inventory planning is to achieve a balance between the cost of holding stocks and the benefits from stock-holding.  
Hence, the technique of inventory planning enables the management to decide when to buy and how much to buy.
6. *OR assists in choosing an optimum strategy.* Game theory is specially used to determine the optimum strategy in a competitive situation and enables businesspeople to maximize profits or minimize losses by adopting the optimum strategy.
7. *OR renders great help in optimum resource allocation.* Linear programming technique is used to allocate scarce resources in an optimum manner in problems of scheduling, product-mix and so on. This technique is popularly used by modern management in resource allocation and in ensuring optimal assignments.
8. *OR facilitates the process of decision-making.* Decision theory enables businessmen to select the best course of action when information is given in a probabilistic form. Through decision tree (a network showing the logical relationship between the different parts of a complex decision and the alternative courses of action in any phase of a decision situation) technique executive's judgement can systematically be brought into the analysis of the problems. Simulation is another important technique used to imitate an operation or process prior to actual performance. The significance of simulation lies in the fact that it enables in finding out the effect of alternative courses of action in a situation involving uncertainty where mathematical formulation is not possible. Even complex groups of variables can be handled through this technique.
9. *Through OR management can know the reactions of the integrated business systems.* The Integrated Production Models technique is used to minimize cost with respect to work force, production and inventory. This technique is quite complex and is usually used by companies having detailed information concerning their sales and costs statistics over a long period. Besides, various

other OR techniques also help management people in taking decisions concerning various problems of business and industry. The techniques are designed to investigate how the integrated business system would react to variations in its component elements and/or external factors.

10. *OR techniques help a lot in the preparation of future (or would be) managers.* In fact, OR techniques substitute a means for improving the knowledge and skill of youngsters in the field of management.

### 1.6.2 Operations Research and Modern Business Management

From what has been stated above, we can say that operational research renders valuable service in the field of business management. It ensures improvement in the quality of managerial decisions in all functional areas of management. The role of OR in business management can be summed up as under:

*OR techniques help the directing authority* in optimum allocation of various limited resources, viz., men, machines, money, material, time, etc., to different competing opportunities on an objective basis for achieving effectively the goal of a business unit. They help the chief of executive in broadening management vision and perspectives in the choice of alternative strategies to the decision problems such as forecasting manpower, production capacities, capital requirements and plans for their acquisition.

*OR is useful to the production management* in (i) Selecting the building site for a plant, scheduling and controlling its development and designing its layout; (ii) Locating within the plant and controlling the movements of required production materials and finished goods inventories; (iii) Scheduling and sequencing production by adequate preventive maintenance with optimum number of operatives by proper allocation of machines; and (iv) Calculating the optimum product-mix.

*OR is useful to the personnel management* to find out (i) Optimum manpower planning; (ii) The number of persons to be maintained on the permanent or full time roll; (iii) The number of persons to be kept in a work pool intended for meeting absenteeism; (iv) The optimum manner of sequencing and routing of personnel to a variety of jobs; and (v) In studying personnel recruiting procedures, accident rates and labour turnover.

*OR techniques equally help the marketing management* to determine (i) Where distribution points and warehousing should be located; their size, quantity to be stocked and the choice of customers; (ii) The optimum allocation of sales budget to direct selling and promotional expenses; (iii) The choice of different media of advertising and bidding strategies; and (iv) The consumer preferences relating to size, colour, packaging, etc., for various products as well as to outbid and outwit competitors.

*OR is also very useful to the financial management* in (i) Finding long-range capital requirements as well as how to generate these requirements; (ii) Determining optimum replacement policies; (iii) Working out a profit plan for the firm; (iv) Developing capital-investments plans; and (v) Estimating credit and investment risks.

In addition to all this, OR provides the business executives such an understanding of the business operations which gives them new insights and capability to determine better solutions for several decision-making problems with great speed, competence and confidence. When applied on the level of management where policies are formulated, OR assists the executives in an advisory capacity but on the operational level where production, personnel, purchasing, inventory and administrative decisions are made. It provides management with a means for handling and processing information. Thus, in brief, OR can be considered as scientific method of providing executive departments with a quantitative basis for taking decisions regarding operations under their control.

## 1.7 FEATURES AND METHODOLOGY OF OPERATIONS RESEARCH AND PHASES OF OPERATIONS RESEARCH STUDY

OR study generally involves three phases viz., the judgement phase, the research phase and the action phase. Of these three, the research phase is the longest and the largest, but the remaining two phases are very important since they provide the basis for and implementation of the research phase respectively.

*The judgement phase* includes (i) A determination of the problem; (ii) The establishment of the objectives and values related to the operation; and (iii) The determination of suitable measures of effectiveness.

*The research phase* utilizes (i) Observations and data collection for better understanding of the problem; (ii) Formulation of hypothesis and models; (iii) Observation and experimentation to test the hypothesis on the basis of additional data; and (iv) predictions of various results from the hypothesis, generalization of the result and consideration of alternative methods.

*The action phase* in the OR consists of making recommendations for decision process. As such this phase deals with the implementation of the tested results of the model. This phase is executed primarily through the cooperation of the OR experts on the one hand and those who are responsible for operating the system on the other.

### 1.7.1 Methodology of Operations Research

In view of the above referred phases the methodology of OR generally involves the following steps:

1. *Formulating the problem.* The first step in an OR study is to formulate the problem in an appropriate form. Formulating a problem consists in identifying, defining and specifying the measures of the components of a decision model. This means that all quantifiable factors which are pertinent to the functioning of the system under consideration are defined in mathematical language as variables (factors which are not controllable), parameters or coefficient along with the constraints on the variables and the determination of suitable measures of effectiveness.
2. *Constructing the model.* The second step consists in constructing the model by which we mean that appropriate mathematical expressions are formulated which

describe interrelations of all variables and parameters. In addition, one or more equations or inequalities are required to express the fact that some or all of the controlled variables can only be manipulated within limits. Such equations or inequalities are termed as constraints or the restrictions. The model must also include an objective function which defines the measure of effectiveness of the system. The objective function and the constraints, together constitute a model of the problem that we want to solve. This model describes the technology and the economics of the system under consideration through a set of simultaneous equations and inequalities.

3. *Deriving the solution.* Once the model is constructed the next step in an OR study is that of obtaining the solution to the model, i.e., finding the optimal values of the controlled variables—values that produce the best performance of the system for specified values of the uncontrolled variables. In other words, an optimum solution is determined on the basis of the various equations of the model satisfying the given constraints and inter-relations of the system and at the same time maximizing profit or minimizing cost or coming as close as possible to some other goal or criterion. How the solution can be derived depends on the nature of the model. In general, there are three methods available for the purpose viz., the analytical methods, the numerical methods and the simulation methods. Analytical methods involve expressions of the model by mathematical computations and the kind of mathematics required depends upon the nature of the model under consideration. This sort of mathematical analysis can be conducted only in some cases without any knowledge of the values of the variables but in others the values of the variables must be known concretely or numerically. In latter cases, we use the numerical methods which are concerned with iterative procedures through the use of numerical computations at each step. The algorithm (or the set of computational rules) is started with a trial or initial solution and continued with a set of rules for improving it towards optimality. The initial solution is then replaced by the improved one and the process is repeated until no further improvement is possible. But in those cases where the analytical as well as the numerical methods cannot be used for deriving the solution then we use simulation methods, i.e., we conduct experiments on the model in which we select values of the uncontrolled variables with the relative frequencies dictated by their probability distributions. The simulation methods involve the use of probability and sampling concepts and are generally used with the help of computers. Whichever method is used, our objective is to find an optimal or near-optimal solution, i.e., a solution which optimizes the measure of effectiveness in a model.
4. *Testing the validity.* The solution values of the model, obtained as stated in step three above, are then tested against actual observations. In other words, effort is made to test the validity of the model used. A model is supposed to be valid if it can give a reliable prediction of the performance of the system represented through the model. If necessary, the model may be modified in the light of actual observations and the whole process is repeated till a satisfactory model is attained. The operational researcher quite often realizes that his model must be a good representation of the system and must correspond to reality which in

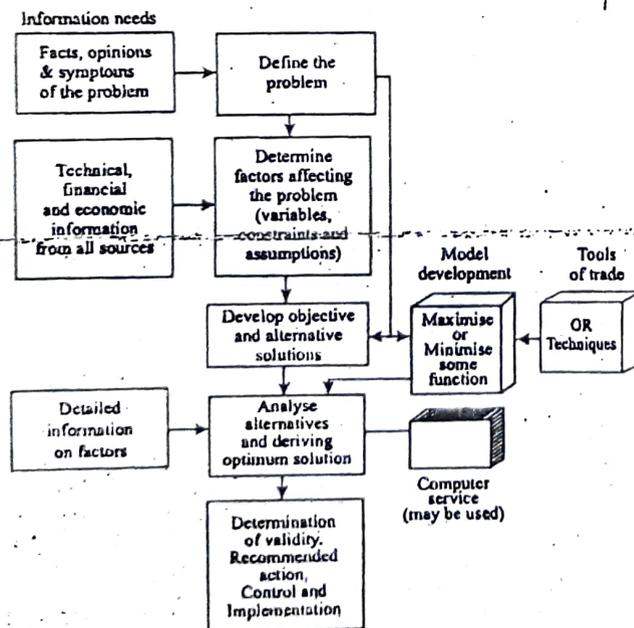
turn requires this step of testing the validity of the model in an OR study. In effect, performance of the model must be compared with the policy or procedure that it is meant to replace.

5. *Controlling the solution.* This step of an OR study establishes control over the solution by proper feedback of information on variables which might have deviated significantly. As such the significant changes in the system and its environment must be detected and the solution must accordingly be adjusted.
6. *Implementing the results.* Implementing the results constitutes the last step of an OR study. The objective of OR is not merely to produce reports but to improve the performance of systems. The results of the research must be implemented if they are accepted by the decision-makers. It is through this step that the ultimate test and evaluation of the research is made and it is in this phase of the study the researcher has the greatest opportunity for learning.

Thus the procedure for an OR study generally involves some major steps viz., formulating the problem, constructing the mathematical model to represent the system under study, deriving a solution from the model, testing the model and the solution so derived, establishing controls over the solution and lastly putting the solution to work-implementation. Although the said phases and the steps are usually initiated in the order listed in an OR study but it should always be kept in mind that they are likely to overlap in time and to interact each phase usually continues until the study is completed.

### Flow Chart Showing OR Approach

OR approach can be well illustrated through the following flow chart:



### Check Your Progress

1. What is the significance of operations research?
2. How does operations research provides solution for various business problems?
3. How operations research is useful to the production management?
4. What all judgement phase includes?
5. What is the first step in OR study?

## 1.8 MODELS IN OPERATIONS RESEARCH AND METHODS OF DERIVING THE SOLUTION

### NOTES

A model in OR is a *simplified representation of an operation or as a process in which only the basic aspects or the most important features of a typical problem under investigation are considered.* The objective of a model is to identify significant factors and interrelationships. The reliability of the solution obtained from a model depends on the validity of the model representing the real system.

A good model must possess the following characteristics:

- (i) It should be capable of taking into account new formulation without having any changes in its frame.
- (ii) Assumptions made in the model should be as small as possible.
- (iii) Variables used in the model must be less in number ensuring that it is simple and coherent.
- (iv) It should be open to parametric type of treatment.
- (v) It should not take much time in its construction for any problem.

### 1.8.1 Advantages of a Model

There are certain significant advantages gained when using a model. These are:

- (i) Problems under consideration become controllable through a model.
- (ii) It provides a logical and systematic approach to the problem.
- (iii) It provides the limitations and scope of an activity.
- (iv) It helps in finding useful tools that eliminate duplication of methods applied to solve problems.
- (v) It helps in finding solutions for research and improvements in a system.
- (vi) It provides an economic description and explanation of either the operation, or the systems they represent.

### 1.8.2 Classification of Models

The classification of models is a subjective problem. They may be distinguished as follows:

- (i) Models by degree of abstraction
- (ii) Models by function
- (iii) Models by structure
- (iv) Models by nature of an environment
- (v) Models by the extent of generality

#### Models by Function

These models consist of (i) Descriptive models, (ii) Predictive models, and (iii) Normative or optimization models.

*Descriptive and Predictive models.* These models describe and predict facts and relationships among the various activities of the problem. These models do not have an objective function as a part of the model to evaluate decision alternatives. In

this model, it is possible to get information as to how one or more factors change as a result of changes in other factors.

**Normative or Optimization Models.** They are prescriptive in nature and develop objective decision-rule for optimum solutions.

### Models by Structure

These models are represented by (i) Iconic or Physical models, (ii) Analog models, and (iii) Mathematic or Symbolic models.

**Iconic or Physical Models.** These are pictorial representations of real systems and have the appearance of the real thing. An iconic model is said to be scaled down or scaled up according to the dimensions of the model which may be smaller or greater than that of the real item, e.g., city maps, blue prints of houses, globe and so on. These models are easy to observe and describe but are difficult to manipulate and are not very useful for the purposes of prediction.

**Analog Models.** These are abstract than the iconic ones. There is no look alike correspondence between these models and real life items. The models in which one set of properties is used to represent another set of properties are called analog models. After the problem is solved, the solution is reinterpreted in terms of the original system. These models are less specific, less concrete but easier to manipulate than iconic models.

**Mathematic or Symbolic Models.** These are most abstract in nature in comparison to others. They employ a set of mathematical symbols to represent the components of the real system. These variables are related together by means of mathematical equations to describe the behaviour of the system. The solution of the problem is then obtained by applying well developed mathematical techniques to the model. The symbolic model is usually the easiest to manipulate experimentally and it is the most general and abstract. Its function is more explanatory than descriptive.

### Models by Nature of an Environment

These models can be classified into (i) Deterministic models, and (ii) Probabilistic or Stochastic models.

**Deterministic Models.** In these models, all parameters and functional relationships are assumed to be known with certainty when the decision is to be made. Linear programming and break even models are the examples of deterministic models.

**Probabilistic or Stochastic Models.** These models are those in which atleast one parameter or decision variable is a random variable. These models reflect to some extent the complexity of the real world and the uncertainty surrounding it.

### Models by the Extent of Generality

These models can be categorized into (i) Specific models, and (ii) General models.

**Specific models.** When a model presents a system at some specific time, it is known as a *specific model*. In these models, if the time factor is not considered then they are termed as *static models*. An inventory problem of determining economic order quantity for the next period assuming that the demand in the planning period would remain same as that of today is an example of static model. *Dynamic programming* may be considered as an example of *dynamic model*.

**General Models.** Simulation and heuristic models fall under the category of general models. These models are used to explore alternative strategies which have been overlooked previously.

## 1.9 LIMITATIONS OF OPERATIONS RESEARCH

OR though is a great aid to management as outlined above but still it cannot be a substitute for decision-making. The choice of a criterion as to what is actually best for a business enterprise is still that of an executive who has to fall back upon his experience and judgement. This is so because of the several limitations of OR. Important limitations are given below:

1. **The inherent limitations concerning mathematical expressions.** OR involves the use of mathematical models, equations and similar other mathematical expressions. Assumptions are always incorporated in the derivation of an equation or model and such an equation or model may be correctly used for the solution of the business problems when the underlying assumptions and variables in the model are present in the concerned problem. If this caution is not given due care then there always remains the possibility of wrong application of OR techniques. Quite often the operations researchers have been accused of having many solutions without being able to find problems that fit.
2. **High costs are involved in the use of OR techniques.** OR techniques usually prove very expensive. Services of specialized persons are invariably called for and along with it the use of computer and its maintenance is also considered while using OR techniques. Hence, only big concerns can think of using such techniques. Even in big business organizations we can expect that OR techniques will continue to be of limited use simply because they are not in many cases worth their cost. As opposed to this a typical manager, exercising intuition and judgement, may be able to make a decision very inexpensively. Thus, the use of OR is a costlier affair and this constitutes an important limitation of OR.
3. **OR does not take into consideration the intangible factors, i.e., non-measurable human factors.** OR makes no allowance for intangible factors such as skill, attitude, vigour of the management in taking decisions but in many instances success or failure hinges upon the consideration of such non-measurable intangible factors. There cannot be any magic formula for getting an answer to management problems but it depends upon proper managerial attitudes and policies.
4. **OR is only a tool of analysis and not the complete decision-making process.** It should always be kept in mind that OR alone cannot make the final decision. It is just a tool and simply suggests best alternatives. In the final analysis many business decisions will involve human element. Thus, OR is at best a supplement to rather than a substitute for management; subjective judgement is likely to remain a principal approach to decision-making.
5. **Other limitations.** Among other limitations of OR, the following deserve mention:

NOTES

NOTES

- (i) *Bias.* The operational researchers must be unbiased. An attempt to shoe-horn results into a confirmation of management's prior preferences can greatly increase the likelihood of failure.
- (ii) *Inadequate objective functions.* The use of a single objective function is often an insufficient basis for decisions. Laws, regulations, public relations, market strategies, etc., may all serve to overrule a choice arrived at in this way.
- (iii) *Internal resistance.* The implementation of an optimal decision may also confront internal obstacles such as trade unions or individual managers with strong preferences for other ways of doing the job.
- (iv) *Competence.* Competent OR analysis calls for the careful specification of alternatives, a full comprehension of the underlying mathematical relationships and a huge mass of data. Formulation of an industrial problem to an OR set programme is quite often a difficult task.
- (v) *Reliability of the prepared solution.* At times a non-linear relationship is changed to linear for fitting the problem to the LP pattern. This may disturb the solution.

- the integrated business systems. The Integrated Production Models technique is used to minimize cost with respect to work force, production and inventory.
- OR provides the business executives such an understanding of the business operations which gives them new insights and capability to determine better solutions for several decision-making problems with great speed, competence and confidence.
- OR study generally involves three phases viz., the judgement phase, the research phase and the action phase.
- The procedure for an OR study generally involves some major steps viz., formulating the problem, constructing the mathematical model to represent the system under study, deriving a solution from the model, testing the model and the solution so derived, establishing controls over the solution and putting the solution to work implementation.
- OR is generally concerned with problems that are tactical rather than strategic in nature.
- A model in OR is a simplified representation of an operation or a process in which only the basic aspects or the most important features of a typical problem under investigation are considered.

### 1.10 SUMMARY

### 1.11 KEY TERMS

- The term operations research (OR) was first coined in 1940 by J.F. McCloskey and F.N. Trefethen.
- OR is a scientific method of providing executive departments with a quantitative basis for decisions regarding the operations under their control.
- OR can be considered as being the application of scientific method by interdisciplinary teams to problems involving the control of organized (man-machine) systems to provide solutions which best serve the purpose of the organization as a whole.
- OR emphasizes on the overall approach to the system. This characteristic of OR is often referred as system oriented.
- OR involves scientific and systematic attack of complex problems to arrive at the optimum solution.
- Linear programming technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost minimization under certain constraints.
- In OR, waiting line or queuing theory deals with mathematical study of queues which are formed whenever the current demand for service exceeds the current capacity to provide that service.
- In OR, decision theory concerns with making sound decisions under conditions of certainty, risk and uncertainty.
- OR techniques are being used in production, procurement, marketing, finance and other allied fields. Through OR, management can know the reactions of

- **Operations research:** The application of scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.
- **Model:** A simplified representation of an operation or a process that considers only the basic aspects or the most important features of a typical problem under investigation with an objective to identify significant factors and their interrelationships
- **Descriptive model:** A type of model that describes facts and relationships among the various activities of the problem. This model collects information on factors that change as a result of changes in other factors
- **Normative or optimization models:** These models are prescriptive in nature and develop objective decision-rule for optimum solutions

Check Your Progress

What is an OR model? What is its objective?

How are the models classified?

Define descriptive models.

Define deterministic models.

Is only a tool of analysis and not the complete decision-making process?

## NOTES

- **Iconic or physical models:** Pictorial representations of real systems that have the appearance of the real thing. Examples are city maps, blue prints of houses, globes, etc
- **Analog models:** These are abstract models and one set of properties is used to represent another set of properties
- **Mathematic or symbolic models:** These models are most abstract in nature and employ a set of mathematical symbols to represent the components of the real system

### 1.12 ANSWERS TO 'CHECK YOUR PROGRESS'

1. The concept of Operations Research came into existence in a military context during World War II, when military management wanted to arrive at decisions on optimal utilization of scarce military resources with a new approach to the systematic and scientific study of the operations of the system.
2. OR is the scientific knowledge through interdisciplinary team effort for the purpose of determining the best utilization of limited resources.
3. Essential characteristics of operations research are:
  - (i) System orientation
  - (ii) Use of interdisciplinary terms
  - (iii) Application of scientific methods
  - (iv) Uncovering new problems
  - (v) Quantitative solutions
  - (vi) Human factors
4. Following are the areas where concept of OR is applied:
  - (i) Agriculture
  - (ii) Finance
  - (iii) Industry
  - (iv) Marketing
  - (v) Personnel Management
  - (vi) Production Management
  - (vii) Research and Development
5. Major phases involved in the application of OR are:
  - (i) Formulating the problem.
  - (ii) Constructing a mathematical model
  - (iii) Deriving the solution from the model
  - (iv) Testing the model and its solution (updating the model)
  - (v) Controlling the solution
  - (vi) Implementation
6. The subject of Operational Research (OR) was developed in military context during World War II, pioneered by the British scientists.
7. This technique is used in finding a solution for optimizing a given objective, such as profit maximization or cost minimization under certain constraints.

This technique is primarily concerned with the optimal allocation of limited resources for optimizing a given function.

8. **Waiting line or queuing theory** deals with mathematical study of queues. The queues are formed whenever the current demand for service exceeds the current capacity to provide that service. Waiting line technique concerns itself with the random arrival of customers at a service station where the facility is limited.
9. **Game theory** is used to determine the optimum strategy in a competitive situation.
10. It refers to a systematic search for optimal solutions to problems that involve many highly complex interrelations that are, moreover, sensitive to multistage effects such as successive time phases.
11. Operations Research has gained increasing importance since World War II in the technology of business and industry administration. It greatly helps in tackling the intricate and complex problems of modern business and industry.
12. OR techniques are being used in production, procurement, marketing, finance and other allied fields. It can be used to solve problems like how best can managers and executives allocate the available resources to various products so that in a given time the profits are maximum or the cost is minimum
13. OR is useful to the production management in
  - (i) Selecting the building site for a plant, scheduling and controlling its development and designing its layout
  - (ii) Locating within the plant and controlling the movements of required production materials and finished goods inventories
  - (iii) Scheduling and sequencing production by adequate preventive maintenance with optimum number of operatives by proper allocation of machines
  - (iv) Calculating the optimum product-mix.
14. The judgement phase includes
  - (i) A determination of the problem
  - (ii) The establishment of the objectives and values related to the operation
  - (iii) The determination of suitable measures of effectiveness.
15. The first step in an OR study is to formulate the problem in an appropriate form. Formulating a problem consists in identifying, defining and specifying the measures of the components of a decision model.
16. A model in OR is a simplified representation of an operation or as a process in which only the basic aspects or the most important features of a typical problem under investigation are considered. The objective of a model is to identify significant factors and interrelationships.
17. The classification of models is a subjective problem. They may be distinguished as follows:
  - (i) Models by degree of abstraction
  - (ii) Models by function

(iii) Models by structure

(iv) Models by nature of an environment

(v) Models by the extent of generality

18. Descriptive models describe facts and relationships among the various activities of the problem. These models do not have an objective function as a part of the model to evaluate decision alternatives. In this model, it is possible to get information as to how one or more factors change as a result of changes in other factors.
19. In deterministic models, all parameters and functional relationships are assumed to be known with certainty when the decision is to be made. Linear programming and break even models are the examples of deterministic models.
20. It should always be kept in mind that OR alone cannot make the final decision. It is just a tool and simply suggests best alternatives. In the final analysis many business decisions will involve human element. Thus, OR is at best a supplement to rather than a substitute for management; subjective judgement is likely to remain a principal approach to decision-making.

### 1.13 QUESTIONS AND EXERCISES

#### Short-Answer Questions

- Where did the concept of operations research originate?
- Name the fields where operations research can be used?
- What do you mean by nature of operations research?
- Which Indian companies use operations research?
- What is inventory control? What is its importance?
- What is decision theory?
- What do you understand by network analysis?
- What is non-linear programming?
- What is integer programming?
- Describe heuristic programming and algorithmic programming. When are these used?
- What is search theory? In which situation is it applied?
- What do you understand by theory of replacement? What is its significance?
- How operations research enables proper deployment of resources?
- How operations research techniques help in directing authority?
- Define research phase.
- What do you mean by testing the validity?
- How competitive problems arise?

18. What is a model in operations research?

19. Write any one limitation of operations research.

#### Long-Answer Questions

- Explain the meaning and origin of operations research with the help of definitions and examples.
- Discuss about the nature of operations research with the help of examples.
- Write an essay on the development of operations research.
- Discuss operations research in India.
- How is operations research used as a tool in decision-making?
- Explain the various types of operations research techniques.
- How is operations research useful to management?
- Explain the significance of operations research in management process.
- Explain the role of operations research in modern business management.
- Explain the various phases of operations research study.
- Describe operations research approach with the help of a flow chart.
- Explain the various categories under which operations research problems are classified.
- How can you classify operations research models? Explain each type with the help of an example.
- What should be the characteristics of a good operations research model? Explain.
- What are the advantages of using a model? Explain and also give your own opinion.
- Write about application of operations research in modern business environment. Explain with suitable examples.
- Explain the limitations of operations research with the help of suitable examples.

### 1.14 FURTHER READING

- Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.
- Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.
- Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.
- Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.
- Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.

# UNIT 2 LINEAR PROGRAMMING, DUALITY AND SENSITIVITY ANALYSIS

## Structure

- 2.0 Introduction
- 2.1 Unit Objectives
- 2.2 Linear Programming: Meaning and Uses
  - 2.2.1 Meaning of Linear Programming
  - 2.2.2 Fields Where Linear Programming can be Used
- 2.3 Concepts, Notations and General Form of Linear Programming Model
  - 2.3.1 Basic Concepts and Notations
  - 2.3.2 General Form of the Linear Programming Model
- 2.4 Applications and Limitations of Linear Programming Problems
- 2.5 Formulation of Linear Programming Problem
  - 2.5.1 Graphic Solution
  - 2.5.2 General Formulation of Linear Programming Problem
  - 2.5.3 Matrix Form of Linear Programming Problem
- 2.6 Solution of Linear Programming Problem: Graphical Solution and Simplex Method
  - 2.6.1 Graphical Solution
  - 2.6.2 Some Important Definitions
  - 2.6.3 Canonical or Standard Forms of LPP
  - 2.6.4 Simplex Method
  - 2.6.5 *M*Method
- 2.7 Duality
- 2.8 Sensitivity Analysis
- 2.9 Summary
- 2.10 Key Terms
- 2.11 Answers to 'Check Your Progress'
- 2.12 Questions and Exercises
- 2.13 Further Reading

## 2.0 INTRODUCTION

In this unit, you will learn the use of linear programming in decision-making. For a manufacturing process, a production manager has to take decisions as to what quantities and which process or processes are to be used so that the cost is minimum and profit is maximum. Currently, this method is used in solving a wide range of practical business problems. The word 'linear' means that the relationships are represented by straight lines. The word 'programming' means following a method for taking decisions systematically.

You will learn the extensive use of Linear Programming (LP) in solving resource allocation problems, production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, etc. Linear programming has been successfully applied in agricultural and industrial applications.

You will also learn a few basic terms like linearity, process and its level, criterion function, constraints, feasible solutions, optimum solution, etc. The term linearity

implies straight line or proportional relationships among the relevant variables. Process means the combination of one or more inputs to produce a particular output. Criterion function is an objective function which is to be either maximized or minimized. Constraints are limitations under which one has to plan and decide. There are restrictions imposed upon decision variables. Feasible solutions are all those possible solutions considering given constraints. An optimum solution is considered the best among feasible solutions.

You will also learn to formulate linear programming problems and put these in a matrix form. The objective function, the set of constraints and the non-negative constraint together form a linear programming problem. In this unit, you will also learn the methods of solving a Linear Programming Problem (LPP) with two decision variables using the graphical method. All linear programming problems may not have unique solutions. You may find some linear programming problems that have an infinite number of optimal solutions, unbounded solutions or even no solution.

Finally, in this unit, you will learn about the canonical or standard form of LPP and duality. In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, the right hand side (RHS) of each constraint and all variables are non-negative. The simplex method and  $M$  method are the methods of solution by iterative procedure in a finite number of steps using matrix. You will also learn about duality and sensitivity analysis.

## 2.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Understand linear programming
- Know the terms associated with a linear programming problem
- Learn how to formulate a linear programming problem
- Form a matrix of a linear programming problem
- Explain the applications and limitations of linear programming problems
- Solve a linear programming problem with two variables using the graphical method
- Describe linear programming problems in canonical form
- Solve linear programming problems using the simplex method
- Solve linear programming problems using the  $M$  method
- Describe duality and solve dual problems
- Understand the basics of sensitivity analysis

## 2.2 LINEAR PROGRAMMING: MEANING AND USES

Decision-making has always been very important in the business and industrial world, particularly with regard to the problems concerning production of commodities. Which commodity/commodities to produce, in what quantities and by which process or processes, are the main questions before a production manager. English economist

Alfred Marshall pointed out that the businessman always studies his production function and his input prices and substitutes one input for another till his costs become the minimum possible. All this sort of substitution, in the opinion of Marshall, is being done by businessman's trained instinct rather than with formal calculations. But now there does exist a method of formal calculations often termed as Linear Programming. This method was first formulated by a Russian mathematician L.V. Kantorovich, but it was developed later in 1947 by George B. Dantzig 'for the purpose of scheduling the complicated procurement activities of the United States Air Force'. Today, this method is being used in solving a wide range of practical business problems. The advent of electronic computers has further increased its applications to solve many other problems in industry. It is being considered as one of the most versatile management tools.

### 2.2.1 Meaning of Linear Programming

Linear Programming (LP) is a major innovation since World War II in the field of business decision-making, particularly under conditions of certainty. The word 'Linear' means that the relationships are represented by straight lines, i.e., the relationships are of the form  $y = a + bx$  and the word 'Programming' means taking decisions systematically. Thus, LP is a decision-making technique under given constraints on the assumption that the relationships amongst the variables representing different phenomena happen to be linear. In fact, Dantzig originally called it 'programming of interdependent activities in a linear structure' but later shortened it to 'Linear Programming'. LP is generally used in solving maximization (sales or profit maximization) or minimization (cost minimization) problems subject to certain assumptions. Putting in a formal way, 'Linear Programming is the maximization (or minimization) of a linear function of variables subject to a constraint of linear inequalities.' Hence, LP is a mathematical technique designed to assist the organization in optimally allocating its available resources under conditions of certainty in problems of scheduling, product-mix and so on.

### 2.2.2 Fields Where Linear Programming can be Used

The problem for which LP provides a solution may be stated to maximize or minimize for some dependent variable which is a function of several independent variables when the independent variables are subject to various restrictions. The dependent variable is usually some economic objectives, such as profits, production, costs, work weeks, tonnage to be shipped, etc. More profits are generally preferred to less profits and lower costs are preferred to higher costs. Hence, it is appropriate to represent either maximization or minimization of the dependent variable as one of the firm's objective. LP is usually concerned with such objectives under given constraints with linearity assumptions. In fact, it is powerful to take in its stride a wide range of business applications. The applications of LP are numerous and are increasing every day. LP is extensively used in solving resource allocation problems. Production planning and scheduling, transportation, sales and advertising, financial planning, portfolio analysis, corporate planning, etc., are some of its most fertile application areas. More specifically, LP has been successfully applied in the following fields:

- (i) *Agricultural applications*: LP can be applied in farm management problems as it relates to the allocation of resources such as acreage, labour, water supply or working capital in such a way that it maximizes net revenue.

NOTES

- (ii) *Contract awards:* Evaluation of tenders by recourse to LP guarantees that the awards are made in the cheapest way.
- (iii) *Industrial applications:* Applications of LP in business and industry are of most diverse kind. Transportation problems concerning cost minimization can be solved by this technique. The technique can also be adopted in solving the problems of production (product-mix) and inventory control.

Thus, LP is the most widely used technique of decision-making in business and industry in modern times in various fields as stated above.

### 2.3 CONCEPTS, NOTATIONS AND GENERAL FORM OF LINEAR PROGRAMMING MODEL

The following are the concepts, notations and forms used in a linear programming model:

#### 2.3.1 Basic Concepts and Notations

There are certain basic concepts and notations to be first understood for easy adoption of the LP technique. A brief mention of such concepts is as follows:

1. *Linearity.* The term linearity implies straight line or proportional relationships among the relevant variables. Linearity in economic theory is known as constant returns which means that if the amount of input doubles, the corresponding output and profit are also doubled. Linearity assumption, thus, implies that if two machines and two workers can produce twice as much as one machine and one worker; four machines and four workers twice as much as two machines and two workers and so on.
2. *Process and its level.* Process means the combination of particular inputs to produce a particular output. In a process, factors of production are used in fixed ratios, of course, depending upon technology and as such no substitution is possible with a process. There may be many processes open to a firm for producing a commodity and one process can be substituted for another. There is, thus, no interference of one process with another when two or more processes are used simultaneously. If a product can be produced in two different ways, then there are two different processes (or activities or decision variables) for the purpose of a linear programme.
3. *Criterion function.* Criterion function is also known as objective function which states the determinants of the quantity either to be maximized or to be minimized. For example, revenue or profit is such a function when it is to be maximized or cost is such a function when the problem is to minimize it. An objective function should include all the possible activities with the revenue (profit) or cost coefficients per unit of production or acquisition. The goal may be either to maximize this function or to minimize this function. In symbolic form, let  $ZX$  denote the value of the objective function at the  $X$  level of the activities included in it. This is the total sum of individual activities produced at a specified level. The activities are denoted as  $j = 1, 2, \dots, n$ . The revenue or cost coefficient of the

$j$ th activity is represented by  $C_j$ . Thus,  $2X_j$ , implies that  $X$  units of activity  $j = 1$  yields a profit (or loss) of  $C_j = 2$ .

4. *Constraints or inequalities.* These are the limitations under which one has to plan and decide, i.e., restrictions imposed upon decision variables. For example, a certain machine requires one worker to be operated upon; another machine requires at least four workers (i.e.,  $> 4$ ); there are at most 20 machine hours (i.e.,  $< 20$ ) available; the weight of the product should be say 10 lbs and so on, are all examples of constraints or why are known as inequalities. Inequalities like  $X > C$  (reads  $X$  is greater than  $C$  or  $X < C$  (reads  $X$  is less than  $C$ )) are termed as strict inequalities. The constraints may be in form of weak inequalities like  $X \leq C$  (reads  $X$  is less than or equal to  $C$ ) or  $X \geq C$  (reads  $C$  is greater than or equal to  $C$ ). Constraints may be in the form of strict equalities like  $X = C$  (reads  $X$  is equal to  $C$ ).

Let  $b_i$  denote the quantity  $b$  of resource  $i$  available for use in various production processes. The coefficient attached to resource  $i$  is the quantity of resource  $i$  required for the production of one unit of product  $j$ .

5. *Feasible solutions.* Feasible solutions are all those possible solutions which can be worked upon under given constraints. The region comprising of all feasible solutions is referred as *Feasible Region*.
6. *Optimum solution.* Optimum solution is the best of the feasible solutions.

#### 2.3.2 General Form of the Linear Programming Model

Linear Programming problem mathematically can be stated as under:

Choose the quantities,

$$X_j \geq 0 \quad (j = 1, \dots, n) \quad \dots(2.1)$$

This is also known as the non-negativity condition and in simple terms means that no  $X$  can be negative.

To maximize,

$$Z = \sum_{j=1}^n C_j X_j \quad \dots(2.2)$$

Subject to the constraints,

$$\sum_{j=1}^n a_{ij} X_j \leq b_i \quad (i = 1, \dots, m) \quad \dots(2.3)$$

The above is the usual structure of a linear programming model in the simplest possible form. This model can be interpreted as a profit maximization situation where  $n$  production activities are pursued at level  $X_j$  which have to be decided upon, subject to a limited amount of  $m$  resources being available. Each unit of the  $j$ th activity yields a return  $C$  and uses an amount  $a_{ij}$  of the  $i$ th resource.  $Z$  denotes the optimal value of the objective function for a given system.

#### Assumptions or the Conditions to be Fulfilled Underlying the LP Model

LP model is based on the assumptions of proportionality, additivity, certainty, continuity and finite choices.

**Proportionality** is assumed in the objective function and the constraint inequalities. In economic terminology this means that there are constant returns to scale, i.e., if one unit of a product contributes ₹ 5 toward profit, then 2 units will contribute ₹ 10, 4 units ₹ 20 and so on.

**Certainty** assumption means the prior knowledge of all the coefficients in the objective function, the coefficients of the constraints and the resource values. LP model operates only under conditions of certainty.

**Additivity** assumption means that the total of all the activities is given by the sum total of each activity conducted separately. For example, the total profit in the objective function is equal to the sum of the profit contributed by each of the products separately.

**Continuity** assumption means that the decision variables are continuous. Accordingly the combinations of output with fractional values, in case of product-mix problems, are possible and obtained frequently.

**Finite choices** assumption implies that finite number of choices are available to a decision-maker and the decision variables do not assume negative values.

## 2.4 APPLICATIONS AND LIMITATIONS OF LINEAR PROGRAMMING PROBLEMS

The applications of linear programming problems are based on linear programming matrix coefficients and data transmission prior to solving the simplex algorithm. The problem can be formulated from the problem statement using linear programming techniques. The following are the objectives of linear programming:

- Identify the objective of the linear programming problem, i.e., which quantity is to be optimized. For example, maximize the profit.
- Identify the decision variables and constraints used in linear programming, for example, production quantities and production limitations are taken as decision variables and constraints.
- Identify the objective functions and constraints in terms of decision variables using information from the problem statement to determine the proper coefficients.
- Add implicit constraints, such as non-negative restrictions.
- Arrange the system of equations in a consistent form and place all the variables on the left side of the equations.

### Applications of Linear Programming

Linear programming problems are associated with the efficient use of allocation of limited resources to meet desired objectives. A solution required to solve the linear programming problem is termed as optimal solution. The linear programming problems contain a very special subclass and depend on mathematical model or description. It is evaluated using relationships and are termed as straight-line or linear. The following are the applications of linear programming:

- Transportation problem
- Diet problem
- Matrix games
- Portfolio optimization
- Crew scheduling

Linear programming problem may be solved using a simplified version of the simplex technique called transportation method. Because of its major application in solving problems involving several product sources and several destinations of products, this type of problem is frequently called the transportation problem. It gets its name from its application to problems involving transporting products from several sources to several destinations. The formation is used to represent more general assignment and scheduling problems as well as transportation and distribution problems. The two common objectives of such problems are:

- To minimize the cost of shipping  $m$  units to  $n$  destinations.
- To maximize the profit of shipping  $m$  units to  $n$  destinations.

The goal of the diet problem is to find the cheapest combination of foods that will satisfy all the daily nutritional requirements of a person. The problem is formulated as a linear program where the objective is to minimize cost and meet constraints which require that nutritional needs be satisfied. The constraints are used to regulate the number of calories and amounts of vitamins, minerals, fats, sodium and cholesterol in the diet.

Game method is used to turn a matrix game into a linear programming problem. It is based on the Min-Max theorem which suggests that each player determines the choice of strategies on the basis of a probability distribution over the player's list of strategies.

The portfolio optimization template calculates the optimal capital of investments that gives the highest return for the least risk. The unique design of the portfolio optimization technique helps in financial investments or business portfolios. The optimization analysis is applied to a portfolio of businesses to represent a desired and beneficial framework for driving capital allocation, investment and divestment decisions.

Crew scheduling is an important application of linear programming problem. It helps if any airline has a problem related to a large potential crew schedules variables. Crew scheduling models are a key to airline competitive cost advantage these days because crew costs are the second largest flying cost after fuel costs.

### Limitations of Linear Programming Problems

Linear programming is applicable if constraints and objective functions are linear, but there are some limitations of this technique which are as follows:

- All the uncertain factors, such as weather conditions, growth rate of industry, etc, are not taken into consideration.
- Integer values are not taken as the solution, e.g., a value is required for fraction and the nearest integer is not taken for the optimal solution.
- Linear programming technique gives those practical-valued answers that are really not desirable with respect to linear programming problem.

NOTES

- It deals with one single objective in real life problem which is more limited and the problems come with multi-objective.
- In linear programming, coefficients and parameters are assumed as constants but in reality they do not take place.
- Blending is a frequently encountered problem in linear programming. For example, if different commodities are purchased which have different characteristics and costs, then the problem helps to decide how much of each commodity would be purchased and blended within specified bound so that the total purchase cost is minimized.

### 2.5 FORMULATION OF LINEAR PROGRAMMING PROBLEM

A linear programming is a mathematical method for determining method to achieve the best outcome, i.e., maximum profit at lowest cost.

#### 2.5.1 Graphic Solution

The procedure for mathematical formulation of an LPP consists of the following steps:

- Step 1:** The decision variables of the problem are noted.
- Step 2:** The objective function to be optimized (maximized or minimized) as a linear function of the decision variables is formulated.
- Step 3:** The other conditions of the problem such as resource limitation, market constraints, interrelations between variables, etc., are formulated as linear inequations or equations in terms of the decision variables.
- Step 4:** The non-negativity constraint from the considerations is added so that the negative values of the decision variables do not have any valid physical interpretation.

The objective function, the set of constraints and the non-negative constraint together form a linear programming problem.

#### 2.5.2 General Formulation of Linear Programming Problem

The general formulation of the LPP can be stated as follows:

In order to find the values of  $n$  decision variables  $X_1, X_2, \dots, X_n$  to maximize or minimize the objective function.

$$Z = C_1X_1 + C_2X_2 + \dots + C_nX_n \quad \dots (2.4)$$

$$\left. \begin{aligned} a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n (\leq, =, \geq) b_1 \\ a_{21}X_1 + a_{22}X_2 + \dots + a_{2n}X_n (\leq, =, \geq) b_2 \\ \vdots \\ a_{m1}X_1 + a_{m2}X_2 + \dots + a_{mn}X_n (\leq, =, \geq) b_m \end{aligned} \right\} \quad \dots (2.5)$$

Here, the constraints can be inequality  $\leq$  or  $\geq$  or even in the form an equation ( $=$ ) and finally satisfy the non-negative restrictions:

$$X_1 \geq 0, X_2 \geq 0 \dots X_n \geq 0 \quad \dots (2.6)$$

### 2.5.3 Matrix Form of Linear Programming Problem

The LPP can be expressed in the matrix form as follows:

Maximize or minimize  $Z = CX \rightarrow$  Objective function

Subject to  $AX (\leq, =, \geq) B \rightarrow$  Constant equation

$B > 0, X \geq 0 \rightarrow$  Non-negativity restrictions

Where,  $X = (X_1, X_2 \dots X_n)$

$$C = (C_1, C_2 \dots C_n)$$

$$B = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix} \quad A = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \dots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{pmatrix}$$

**Example 2.1:** A manufacturer produces two types of models  $M_1$  and  $M_2$ . Each model of the type  $M_1$  requires 4 hours of grinding and 2 hours of polishing; whereas each model of the type  $M_2$  requires 2 hours of grinding and 5 hours of polishing. The manufacturers have 2 grinders and 3 polishers. Each grinder works 40 hours a week and each polisher works for 60 hours a week. The profit on  $M_1$  model is ₹ 3.00 and on model  $M_2$  is ₹ 4.00. Whatever is produced in a week is sold in the market. How should the manufacturer allocate his production capacity to the two types of models, so that he may make the maximum profit in a week?

**Solution:**

**Decision variables:** Let  $X_1$  and  $X_2$  be the number of units of  $M_1$  and  $M_2$ .

**Objective function:** Since the profit on both the models are given, we have to maximize the profit, viz.,

$$\text{Max } Z = 3X_1 + 4X_2$$

**Constraints:** There are two constraints: one for grinding and the other for polishing.

The number of hours available on each grinder for one week is 40 hours. There are 2 grinders. Hence, the manufacturer does not have more than  $2 \times 40 = 80$  hours for grinding.  $M_1$  requires 4 hours of grinding and  $M_2$  requires 2 hours of grinding.

The grinding constraint is given by,

$$4X_1 + 2X_2 \leq 80$$

Since there are 3 polishers, the available time for polishing in a week is given by  $3 \times 60 = 180$ .  $M_1$  requires 2 hours of polishing and  $M_2$  requires 5 hours of polishing. Hence, we have  $2X_1 + 5X_2 \leq 180$

**Check Your Progress**

1. What is linear programming?
2. What is meant by criterion function in linear programming?
3. Mention two areas where linear programming finds application.
4. What are the constraints in linear programming?

Thus, we have,

$$\text{Max } Z = 3X_1 + 4X_2$$

$$\text{Subject to, } 4X_1 + 2X_2 \leq 80$$

$$2X_1 + 5X_2 \leq 180$$

$$X_1, X_2 \geq 0$$

**Example 2.2:** A firm manufactures three products *A*, *B* and *C*. The profits are ₹ 3, ₹ 2 and ₹ 4 respectively. The firm has two machines and the following is the required processing time in minutes for each machine on each product.

		Product		
		A	B	C
Machines	C	4	3	5
	D	3	2	4

Machine *C* and *D* have 2000 and 2500 machine minutes respectively. The firm must manufacture 100 units of *A*, 200 units of *B* and 50 units of *C*, but not more than 150 units of *A*. Set up an LP problem to maximize the profit.

**Solution:** Let  $X_1, X_2, X_3$  be the number of units of the product *A*, *B*, *C* respectively.

Since the profits are ₹ 3, ₹ 2 and ₹ 4 respectively, the total profit gained by the firm after selling these three products is given by,

$$Z = 3X_1 + 2X_2 + 4X_3$$

The total number of minutes required in producing these three products at machine *C* is given by  $4X_1 + 3X_2 + 5X_3$  and at machine *D* is given by,

$$3X_1 + 2X_2 + 4X_3$$

The restrictions on the machine *C* and *D* are given by 2000 minutes and 2500 minutes.

$$4X_1 + 3X_2 + 5X_3 \leq 2000$$

$$3X_1 + 2X_2 + 4X_3 \leq 2500$$

Also, since the firm manufactures 100 units of *A*, 200 units of *B* and 50 units of *C*, but not more than 150 units of *A*, the further restriction becomes,

$$100 \leq X_1 \leq 150$$

$$200 \leq X_2 \leq 200$$

$$50 \leq X_3 \leq 50$$

Hence, the allocation problem of the firm can be finally put in the following form:

Find the value of  $X_1, X_2, X_3$  so as to maximize,

$$Z = 3X_1 + 2X_2 + 4X_3$$

Subject to the constraints,

$$4X_1 + 3X_2 + 5X_3 \leq 2000$$

$$3X_1 + 2X_2 + 4X_3 \leq 2500$$

$$100 \leq X_1 \leq 150, 200 \leq X_2 \leq 200, 50 \leq X_3 \leq 50$$

## 2.6 SOLUTION OF LINEAR PROGRAMMING PROBLEM: GRAPHICAL SOLUTION AND SIMPLEX METHOD

The linear programming problems can be solved as follows using the graphical solution and simplex method:

### 2.6.1 Graphical Solution

Simple linear programming problem with two decision variables can be easily solved by graphical method.

#### Procedure for Solving LPP by Graphical Method

The steps involved in the graphical method are as follows:

**Step 1:** Consider each inequality constraint as an equation.

**Step 2:** Plot each equation on the graph as each will geometrically represent a straight line.

**Step 3:** Mark the region. If the inequality constraint corresponding to that line is  $\leq$ , then the region below the line lying in the first quadrant (due to non-negativity of variables) is shaded. For the inequality constraint  $\geq$  sign, the region above the line in the first quadrant is shaded. The points lying in the common region will satisfy all the constraints simultaneously. The common region thus obtained is called the feasible region.

**Step 4:** Allocate an arbitrary value, say zero, for the objective function.

**Step 5:** Draw the straight line to represent the objective function with the arbitrary value (i.e., a straight line through the origin).

**Step 6:** Stretch the objective function line till the extreme points of the feasible region. In the maximization case, this line will stop farthest from the origin and passes through at least one corner of the feasible region. In the minimization case this line will stop nearest to the origin and passes through at least one corner of the feasible region.

**Step 7:** Find the coordinates of the extreme points selected in Step 6 and find the maximum or minimum value of  $Z$ .

**Note:** As the optimal values occur at the corner points of the feasible region, it is enough to calculate the value of the objective function of the corner points of the feasible region and select the one which gives the optimal solution, i.e., in the case of maximization problem, optimal point corresponds to the corner point at which the objective function has a maximum value and in the case of minimization, the corner point which gives the objective function the minimum value is the optimal solution.

**Example 2.3:** Solve the following LPP by graphical method.

$$\text{Minimize, } Z = 20X_1 + 10X_2$$

$$\text{Subject to, } X_1 + 2X_2 \leq 40$$

$$3X_1 + X_2 \geq 30$$

NOTES

$$4X_1 + 3X_2 \geq 60$$

$$X_1, X_2 \geq 0$$

Solution: Replace all the inequalities of the constraints by equation,

$$X_1 + 2X_2 = 40 \text{ If } X_1 = 0 \Rightarrow X_2 = 20$$

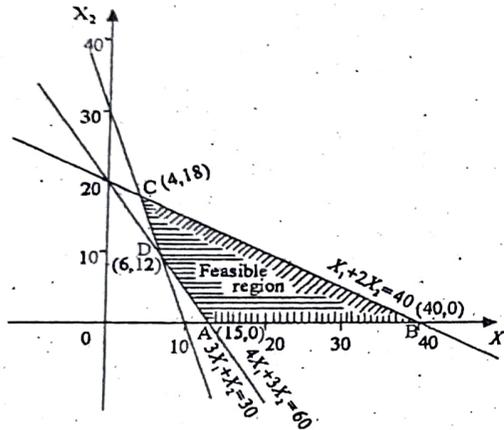
$$\text{If } X_2 = 0 \Rightarrow X_1 = 40$$

$$\therefore X_1 + 2X_2 = 40 \text{ passes through } (0, 20) (40, 0)$$

$$3X_1 + X_2 = 30 \text{ passes through } (0, 30) (10, 0)$$

$$4X_1 + 3X_2 = 60 \text{ passes through } (0, 20) (15, 0)$$

Plot each equation on the graph.



The feasible region is  $ABCD$ .

$C$  and  $D$  are points of intersection of lines.

$$X_1 + 2X_2 = 40, 3X_1 + X_2 = 30$$

$$\text{And, } 4X_1 + 3X_2 = 60$$

On solving, we get  $C(4, 18)$  and  $D(6, 12)$

Corner Points Value of  $Z = 20X_1 + 10X_2$

$$A(15, 0) \quad 300$$

$$B(40, 0) \quad 800$$

$$C(4, 18) \quad 260$$

$$D(6, 12) \quad 240 \text{ (Minimum value)}$$

$\therefore$  The minimum value of  $Z$  occurs at  $D(6, 12)$ . Hence, the optimal solution is

$$X_1 = 6, X_2 = 12.$$

**Example 2.4:** Find the maximum value of  $Z = 5X_1 + 7X_2$

Subject to the constraints,

$$X_1 + X_2 \leq 4$$

$$3X_1 + 8X_2 \leq 24$$

$$10X_1 + 7X_2 \leq 35$$

$$X_1, X_2 > 0$$

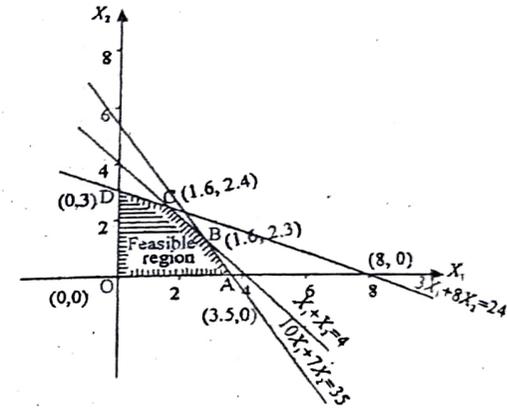
Solution: Replace all the inequalities of the constraints by forming equations.

$$X_1 + X_2 = 4 \text{ passes through } (0, 4) (4, 0)$$

$$3X_1 + 8X_2 = 24 \text{ passes through } (0, 3) (8, 0)$$

$$10X_1 + 7X_2 = 35 \text{ passes through } (0, 5) (3.5, 0)$$

Plot these lines in the graph and mark the region below the line as the inequality of the constraint is  $\leq$  and is also lying in the first quadrant.



The feasible region is  $OABCD$ .

$B$  and  $C$  are points of intersection of lines,

$$X_1 + X_2 = 4, 10X_1 + 7X_2 = 35$$

$$\text{And, } 3X_1 + 8X_2 = 24$$

On-solving we get,

$$B(1.6, 2.3)$$

$$C(1.6, 2.4)$$

Corner Points Value of  $Z = 5X_1 + 7X_2$

$$O(0, 0) \quad 0$$

$$A(3.5, 0) \quad 17.5$$

$$B(1.6, 2.3) \quad 25.1$$

$$C(1.6, 2.4) \quad 24.8 \text{ (Maximum value)}$$

$$D(0, 3) \quad 21$$

$\therefore$  The maximum value of  $Z$  occurs at  $C(1.6, 2.4)$  and the optimal solution is  $X_1 = 1.6, X_2 = 2.4$ .

Example 2.5: Solve the following LPP by graphical method.

$$\text{Maximize, } Z = 100X_1 + 40X_2$$

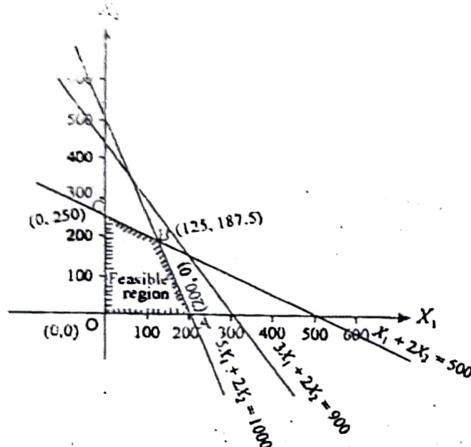
$$\text{Subject to, } 5X_1 + 2X_2 \leq 1000$$

$$3X_1 + 2X_2 \leq 900$$

$$X_1 + 2X_2 \leq 500$$

$$\text{and } X_1, X_2 \geq 0$$

Solution:



The solution space is given by the feasible region  $OABC$ .

Corner Points	Value of $Z = 100X_1 + 40X_2$
$O(0, 0)$	0
$A(200, 0)$	20,000
$B(125, 187.5)$	20,000 (Maximum value of $Z$ )
$C(0, 250)$	10,000

The maximum value of  $Z$  occurs at two vertices  $A$  and  $B$ .

Since there are infinite number of points on the line, joining  $A$  and  $B$  gives the same maximum value of  $Z$ .

Thus, there are infinite number of optimal solutions for the LPP.

### 2.6.2 Some Important Definitions

1. A set of values  $X_1, X_2, \dots, X_n$  which satisfies the constraints of the LPP is called its solution.
2. Any solution to a LPP which satisfies the non-negativity restrictions of the LPP is called its feasible solution.
3. Any feasible solution which optimizes (minimizes or maximizes) the objective function of the LPP is called its optimum solution.
4. Given a system of  $m$  linear equations with  $n$  variables ( $m < n$ ), any solution which is obtained by solving  $m$  variables keeping the remaining  $n - m$  variables

zero is called a basic solution. Such  $m$  variables are called basic variables and the remaining variables are called non-basic variables.

5. A basic feasible solution is a basic solution which also satisfies all basic variables are non-negative.

Basic feasible solutions are of two types:

- (i) *Non-degenerate*: A non-degenerate basic feasible solution is the basic feasible solution which has exactly  $m$  positive  $X_i$  ( $i = 1, 2, \dots, m$ ), i.e., none of the basic variables is zero.
- (ii) *Degenerate*: A basic feasible solution is said to be degenerate if one or more basic variables are zero.

6. If the value of the objective function  $Z$  can be increased or decreased indefinitely, such solutions are called unbounded solutions.

### 2.6.3 Canonical or Standard Forms of LPP

The general LPP can be put in either canonical or standard forms.

In the standard form, irrespective of the objective function, namely maximize or minimize, all the constraints are expressed as equations. Moreover, RHS of each constraint and all variables are non-negative.

#### Characteristics of the Standard Form

- (i) The objective function is of maximization type.
- (ii) All constraints are expressed as equations.
- (iii) Right hand side of each constraint is non-negative.
- (iv) All variables are non-negative.

In the canonical form, if the objective function is of maximization, all the constraints other than non-negativity conditions are ' $\leq$ ' type. If the objective function is of minimization, all the constraints other than non-negative conditions are ' $\geq$ ' type.

#### Characteristics of the Canonical Form

- (i) The objective function is of maximization type.
- (ii) All constraints are of ' $\leq$ ' type.
- (iii) All variables  $X_i$  are non-negative.

Notes:

1. Minimization of a function  $Z$  is equivalent to maximization of the negative expression of this function, i.e.,  $\text{Min } Z = -\text{Max } (-Z)$ .
2. An inequality in one direction can be converted into an inequality in the opposite direction by multiplying both sides by  $(-1)$ .
3. Suppose we have the constraint equation,

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n = b_1$$

This equation can be replaced by two weak inequalities in opposite directions.

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n \leq b_1$$

$$a_{11}X_1 + a_{12}X_2 + \dots + a_{1n}X_n \geq b_1$$

#### Check Your Progress

5. What do you understand by solution in the context of linear programming problem?
6. What is a 'basic solution' of an LPP?
7. What is meant by basic and non-basic variables?
8. What do you understand by basic feasible solution?

## NOTES

- If a variable is unrestricted in sign, then it can be expressed as a difference of two non-negative variables, i.e.,  $X_i$  is unrestricted in sign, then  $X_i = X_i' - X_i''$ , where  $X_i, X_i', X_i''$  are  $\geq 0$ .
- In standard form, all the constraints are expressed in equation, which is possible by introducing some additional variables called slack variables and surplus variables so that a system of simultaneous linear equations is obtained. The necessary transformation will be made to ensure that  $b_i \geq 0$ .

### Definition

(i) If the constraints of a general LPP be,

$$\sum_{j=1}^n a_{ij} X_j \leq b_i \quad (i = 1, 2, \dots, m),$$

then the non-negative variables  $S_i$  which are introduced to convert the inequalities ( $\leq$ ) to the equalities  $\sum_{j=1}^n a_{ij} X_j + S_i = b_i \quad (i = 1, 2, \dots, m)$ , are called slack variables.

Slack variables are also defined as the non-negative variables which are added in the LHS of the constraint to convert the inequality ' $\leq$ ' into an equation.

(ii) If the constraints of a general LPP be,

$$\sum_{j=1}^n a_{ij} X_j \geq b_i \quad (i = 1, 2, \dots, m),$$

then, the non-negative variables  $S_i$  which are introduced to convert the inequalities ' $\geq$ ' to the equalities  $\sum_{j=1}^n a_{ij} X_j - S_i = b_i \quad (i = 1, 2, \dots, m)$  are called surplus variables.

Surplus variables are defined as the non-negative variables which are removed from the LHS of the constraint to convert the inequality ' $\geq$ ' into an equation.

### 2.6.4 Simplex Method

Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be that at the previous vertex. This procedure is repeated and since the number of vertices is finite, the method leads to an optimal vertex in a finite number of steps or indicates the existence of unbounded solution.

### Definition

(i) Let  $X_B$  be a basic feasible solution to the LPP.

$$\text{Max } Z = C_X$$

Subject to  $A_X = b$  and  $X \geq 0$ , such that it satisfies  $X_B = B^{-1}b$ ,

Where  $B$  is the basic matrix formed by the column of basic variables.

The vector  $C_B = (C_{B1}, C_{B2}, \dots, C_{Bm})$ , where  $C_{Bj}$  are components of  $C$  associated with the basic variables is called the cost vector associated with the basic feasible solution  $X_B$ .

(ii) Let  $X_B$  be a basic feasible solution to the LPP.

$$\text{Max } Z = C_X \text{ where } A_X = b \text{ and } X \geq 0.$$

Let  $C_B$  be the cost vector corresponding to  $X_B$ . For each column vector  $a_j$  in  $A$ , which is not a column vector of  $B$ , let

$$a_j = \sum_{i=1}^m a_{ij} b_i$$

Then the number  $Z_j = \sum_{i=1}^m C_{Bi} a_{ij}$  is called the evaluation corresponding to  $a_j$

and the number  $(Z_j - C_j)$  is called the net evaluation corresponding to  $j$ .

### Simplex Algorithm

For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed. The steps for the computation of an optimum solution are as follows:

**Step 1:** Check whether the objective function of the given LPP is to be maximized or minimized. If it is to be minimized then we convert it into a problem of maximization by,

$$\text{Min } Z = -\text{Max } (-Z)$$

**Step 2:** Check whether all  $b_i \quad (i = 1, 2, \dots, m)$  are positive. If any one of  $b_i$  is negative, then multiply the inequation of the constraint by  $-1$  so as to get all  $b_i$  to be positive.

**Step 3:** Express the problem in the standard form by introducing slack/surplus variables to convert the inequality constraints into equations.

**Step 4:** Obtain an initial basic feasible solution to the problem in the form  $X_B = B^{-1}b$  and put it in the first column of the simplex table. Form the initial simplex table shown as follows:

		$C_j$	$C_1$	$C_2$	$C_3$	..	..	0 0 .....	0	
$C_B$	$S_B$	$X_B$	$X_1$	$X_2$	$X_3$	$X_4$	.....	$X_n$	$S_1, S_2, \dots, S_m$	
$C_{B1}$	$S_1$	$b_1$	$a_{11}$	$a_{12}$	$a_{13}$	$a_{14}$	.....	$a_{1n}$	1 0 .....	0
$C_{B2}$	$S_2$	$b_2$	$a_{21}$	$a_{22}$	$a_{23}$	$a_{24}$	.....	$a_{2n}$	1 0 .....	0

**Step 5:** Compute the net evaluations  $Z_j - C_j$  by using the relation:

$$Z_j - C_j = C_B (a_j - C_j)$$

Examine the sign of  $Z_j - C_j$ :

(i) If all  $Z_j - C_j \geq 0$ , then the initial basic feasible solution  $X_B$  is an optimum basic feasible solution.

(ii) If at least one  $Z_j - C_j > 0$ , then proceed to the next step as the solution is not optimal.

**Step 6:** To find the entering variable, i.e., key column.

If there are more than one negative  $Z_j - C_j$  choose the most negative of them. Let it be  $Z_r - C_r$  for some  $j = r$ . This gives the entering variable  $X_r$  and is indicated by an arrow at the bottom of the  $r$ th column. If there are more than one variable having the same most negative  $Z_j - C_j$ , then any one of the variable can be selected arbitrarily as the entering variable.

- (i) If all  $X_{ij} \leq 0$  ( $i = 1, 2, \dots, m$ ) then there is an unbounded solution to the given problem.
- (ii) If at least one  $X_{ij} > 0$  ( $i = 1, 2, \dots, m$ ), then the corresponding vector  $X_i$  enters the basis.

Step 7: To find the leaving variable or key row:

Compute the ratio  $(X_{bi}/X_{br}, X_{br} > 0)$

If the minimum of these ratios be  $X_{bi}/X_{br}$ , then choose the variable  $X_i$  to leave the basis called the key row and the element at the intersection of the key row and the key column is called the key element.

Step 8: Form a new basis by dropping the leaving variable and introducing the entering variable along with the associated value under  $C_b$  column. The leaving element is converted to unity by dividing the key equation by the key element and all other elements in its column to zero by using the formula:

New element = Old element

$$-\left[ \frac{\text{Product of elements in key row and key column}}{\text{Key element}} \right]$$

Step 9: Repeat the procedure of Step (5) until either an optimum solution is obtained or there is an indication of unbounded solution.

Example 2.6: Use simplex method to solve the following LPP.

Maximize,  $Z = 3X_1 + 2X_2$

Subject to,  $X_1 + X_2 \leq 4$

$X_1 - X_2 \leq 2$

$X_1, X_2 \geq 0$

Solution: By introducing the slack variables  $S_1$  and  $S_2$  convert the problem into standard form.

Max,  $Z = 3X_1 + 2X_2 + 0S_1 + 0S_2$

Subject to,  $X_1 + X_2 + S_1 = 4$

$X_1 - X_2 + S_2 = 2$

$X_1, X_2, S_1, S_2 \geq 0$

$$\begin{bmatrix} X_1 & X_2 & S_1 & S_2 \\ 1 & 1 & 1 & 0 \\ 1 & -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} X_1 \\ X_2 \\ S_1 \\ S_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

An initial basic feasible solution is given by,

$X_B = B^{-1}b$ ,

Where,  $B = I_2$ ;  $X_B = (S_1, S_2)$

i.e.,  $(S_1, S_2) = I_2(4, 2) = (4, 2)$

Initial Simplex Table

$Z_j = C_B a_j$

$Z_1 - C_1 = C_B a_1 - C_1 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 \ 1) - 3 = -3$

$Z_2 - C_2 = C_B a_2 - C_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 \ 1) - 2 = -2$

$Z_3 - C_3 = C_B a_3 - C_3 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (1 \ 0) - 0 = -0$

$Z_4 - C_4 = C_B a_4 - C_4 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} (0 \ 1) - 0 = -0$

	$C_j$		3	2	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	$\text{Min } \frac{X_B}{X_1}$
0	$S_1$	4	1	1	1	0	$4/1 = 4$
$\leftarrow 0$	$S_2$	2	Ⓐ	-1	0	1	$2/1 = 2$
	$Z_j$	0	0	0	0	0	
	$Z_j - C_j$		-3↑	-2	0	0	

Since, there are some  $Z_j - C_j = 0$ , the current basic feasible solution is not optimum.

Since,  $Z_1 - C_1 = -3$  is the most negative, the corresponding non-basic variable  $X_1$  enters the basis.

The column corresponding to this  $X_1$  is called the key column.

Ratio =  $\text{Min} \left\{ \frac{X_{bi}}{X_{br}}, X_{br} > 0 \right\}$

=  $\text{Min} \left\{ \frac{4}{1}, \frac{2}{1} \right\}$ , which corresponds to  $S_2$

∴ The leaving variable is the basic variable  $S_2$ . This row is called the key row. Convert the leading element  $X_{21}$  to unity and all other elements in its column  $n$ , i.e.,  $(X_1)$  to zero by using the formula:

New element = Old element -

$$\left[ \frac{\text{Product of elements in key row and key column}}{\text{Key element}} \right]$$

To apply this formula, first we find the ratio, namely

NOTES

$$\frac{\text{The element to be zero}}{\text{Key element}} = \frac{1}{1} = 1$$

Apply this ratio for the number of elements that are converted in the key row.  
Multiply this ratio by key row element shown as follows:

- $1 \times 2$
- $1 \times 1$
- $1 \times -1$
- $1 \times 0$
- $1 \times 1$

Now, subtract this element from the old element. The element to be converted into zero is called the old element row. Finally, we have

- $4 - 1 \times 2 = 2$
- $1 - 1 \times 1 = 0$
- $1 - 1 \times -1 = 2$
- $1 - 1 \times 0 = 1$
- $0 - 1 \times 1 = -1$

∴ The improved basic feasible solution is given in the following simplex table.

First Iteration

	$C_j$	3	2	0	0		
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	Min $\frac{X_B}{X_2}$
←0	$S_1$	2	0	2	1	-1	$2/2 = 1$
3	$X_1$	2	1	-1	0	1	-
	$Z_j$	6	3	-3	0	0	
	$Z_j - C_j$		0	-5↑	0	0	

Since,  $Z_j - C_j$  is the most negative,  $X_2$  enters the basis.

To find Min  $\left( \frac{X_B}{X_{12}}, X_{12} > 0 \right)$

$$\text{Min} \left( \frac{2}{2} \right)$$

This gives the outgoing variables. Convert the leaving element into one. This is done by dividing all the elements in the key row by 2. The remaining elements are converted to zero by using the following formula.

Here,  $-\frac{1}{2}$  is the common ratio. Put this ratio 5 times and multiply each ratio by the key row element.

- $-\frac{1}{2} \times 2$
- $-\frac{1}{2} \times 0$
- $-\frac{1}{2} \times 2$
- $-1/2 \times 1$
- $-1/2 \times -1$

Subtract this from the old element. All the row elements which are converted into zero are called the old elements.

- $2 - \left( -\frac{1}{2} \times 2 \right) = 3$
- $1 - (-1/2 \times 0) = 1$
- $-1 - (-1/2 \times 2) = 0$
- $0 - (-1/2 \times 1) = 1/2$
- $1 - (-1/2 \times -1) = 1/2$

Second Iteration

	$C_j$	3	2	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$
2	$X_2$	1	0	1	1/2	-1/2
3	$X_1$	3	1	0	1/2	1/2
	$Z_j$	11	3	2	5/2	1/2
	$Z_j - C_j$		0	0	5/2	1/2

Since all  $Z_j - C_j \geq 0$ , the solution is optimum. The optimal solution is Max  $Z = 11, X_1 = 3$ , and  $X_2 = 1$ .

Example 2.7: Solve the following LPP using simplex method.

- Maximize  $Z = 3X_1 + 2X_2 + 5X_3$
- Subject to,  $X_1 + 2X_2 + X_3 \leq 430$
- $3X_1 + 2X_2 \leq 460$
- $X_1 + 4X_2 \leq 420$
- $X_1, X_2, X_3 \geq 0$

Solution: Rewrite the constraint into an equation by adding slack variables  $S_1, S_2$  and  $S_3$ . The standard form of LPP becomes,

- Maximize,  $Z = 3X_1 + 2X_2 + 5X_3 + 0S_1 + 0S_2 + 0S_3$
- Subject to,  $X_1 + 2X_2 + X_3 + S_1 = 430$
- $3X_1 + 2X_2 + S_2 = 460$
- $X_1 + 4X_2 + S_3 = 420$
- $X_1, X_2, X_3, S_1, S_2, S_3 \geq 0$

The initial basic feasible solution is,  
 $S_1 = 430, S_2 = 460, S_3 = 420$  ( $X_1 = X_2 = X_3 = 0$ )

Initial Table

	$C_j$	3	2	5	0	0	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$X_3$	$S_1$	$S_2$	$S_3$	Min $\frac{X_B}{X_j}$
0	$S_1$	430	1	2	1	1	0	0	430/1=430
←0	$S_2$	460	3	0	②	0	1	0	460/2=230
0	$S_3$	420	1	4	0	0	0	1	
	$Z_j$	0	0	0	0	0	0	0	
	$Z_j - C_j$		-3	-2	-5↑	0	0	0	

Since some of  $Z_j - C_j \leq 0$ , the current basic feasible solution is not optimum. Since  $Z_3 - C_3 = -5$  is the most negative, the variable  $X_3$  enters the basis. To find the variable leaving the basis, find

$$\text{Min} \left( \frac{X_B}{X_3}, X_B > 0 \right) = \text{Min} \left( \frac{430}{1} = 430, \frac{460}{2} = 230 \right)$$

∴ The variable  $S_2$  leaves the basis.

First Iteration

	$C_j$	3	2	5	0	0	0		
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$X_3$	$S_1$	$S_2$	$S_3$	Min $\frac{X_B}{X_2}$
←0	$S_1$	200	-1/2	②	0	1	1/2	0	200/2=100
5	$X_3$	230	3/2	0	1	0	1/2	0	
0	$S_3$	420	1	4	0	0	0	1	420/4=105
	$Z_j$	1150	15/2	0	0	0	5/2	0	
	$Z_j - C_j$		9/2	-2↑	0	0	5/2	0	

Since  $Z_2 - C_2 = -2$  is negative, the current basic feasible solution is not optimum. Therefore, the variable  $X_2$  enters the basis and the variable  $S_1$  leaves the basis.

### 2.6.5 M Method

In simplex algorithm, the M Method is used to deal with the situation where an infeasible starting basic solution is given. The simplex method starts from one basic feasible solution (BFS) or the intense point of the feasible region of a linear programming problem (LPP) presented in tableau form and extends to another BFS for constantly raising or reducing the value of the objective task till optimality is reached. Sometimes the starting basic solution may be infeasible, then M method is used to find the starting basic feasible solution each time it exists.

**Example 2.8:** Find a starting basic feasible solution each time it exists for the following LPP where there is no starting identity matrix using M method.

Maximize,  $X_0 = C^T X$

Subject to,  $AX = b, X \geq 0$ ; Where  $b > 0$ .

**Solution:** To get a starting identity matrix, we add artificial variables  $X_{a1}, X_{a2}, \dots, X_{am}$ . The consequent values for the artificial variables can be M for maximization problem (where M is adequately large). This constant M will check artificial variables that will arise with positive values in the final optimal solutions. Now the LPP becomes,

$$\text{Max } Z = C^T X - M \cdot 1^T X_0$$

$$\text{Subject to, } AX + I_m X_0 = b,$$

$$X \geq 0$$

Where  $X_0 = (X_{a1}, X_{a2}, \dots, X_{am})^T$  and 1 is the vector of all ones. Here,  $X = 0$  and  $X_0 = b$  is the feasible starting basic feasible solution. For solving  $AX + I_m X_0 = b$ , which is a solution to  $AX = b$  we have to drive and take  $X_0 = 0$ .

**Example 2.9:** Using the linear programming, solve the following LPP:

$$\text{Maximize, } X_0 = X_1 + X_2$$

$$\text{Subject to, } 2X_1 + X_2 \geq 4$$

$$X_1 + 2X_2 = 6$$

$$X_1, X_2 \geq 0$$

**Solution:** Add surplus variable  $X_3$  and artificial variables  $X_4$  and  $X_5$ , and then rewrite the equation as given below:

$$2X_1 + X_2 - X_3 + X_4 = 4$$

$$X_1 + 2X_2 + X_5 = 6$$

$$X_0 - X_1 - X_2 + M X_4 + M X_5 = 0$$

The columns corresponding to  $X_4$  and  $X_5$  form an identity matrix. This can be represented in tableau form as,

	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$b$
$X_4$	2	1	-1	1	0	4
$X_5$	1	2	0	0	1	6
$X_0$	-1	-1	0	M	M	0

In the above table the row  $X_0$  has the reduced cost coefficient for basic variables  $X_4$  and  $X_5$  which are not zero. First eliminate these nonzero entries to have the initial tableau.

	$X_1$	$X_2$	$X_3$	$X_4$	$X_5$	$b$
$X_4$	2	1	-1	1	0	4
$X_5$	1	2	0	0	1	6
$X_0$	$-(1+3M)$	$-(1+3M)$	M	0	0	-10M

The artificial variable becomes non-basic and can be dropped in subsequent calculations. Now the tableau becomes:

NOTES

	$X_1$	$X_2$	$X_3$	$X_4$	$b$
$X_1$	1	1/2	-1/2	0	2
$X_3$	0	3/2	1/2	1	4
$X_0$	0	$-(1+3M)/2$	$-(1+M)/2$	0	$2-4M$

Eliminating artificial variables we get,

	$X_1$	$X_2$	$X_3$	$b$
$X_1$	1	0	-2/3	2/3
$X_2$	0	1	1/3	8/3
$X_0$	0	0	-1/3	10/3

Now all the artificial variables are eliminated and  $X = [2/3, 8/3, 0]^T$  is an initial basic feasible solution. Iterating again we get the following final optimal tableau:

	$X_1$	$X_2$	$X_3$	$b$
$X_1$	1	2	0	6
$X_2$	0	3	1	8
$X_0$	0	1	0	6

Hence, the optimal solution is  $X = (6, 0, 8)^T$  with  $X_0 = 6$ .

2.7 DUALITY

For every given linear programming problem, there is another intimately related linear programming problem referred to as its dual. The duality theorem states that 'for every maximization (or minimization) problem in linear programming, there is a unique similar problem of minimization (or maximization) involving the same data which describes the original problem'. The original problem is referred to as the 'primal'. The 'dual' of a dual problem is the primal. Thus the primal and dual problems are replicas of each other. Further, the maximum feasible value of the primal objective function equals to the minimum feasible value of the dual objective function. This means that the solutions of the primal and the dual problems are related which infact yields several advantages.

The transformation of a given primal problem into a dual problem involves the following considerations:

- (1) If the objective of the primal is maximization, the objective of the dual is minimization.
- (2) The primal has  $m$ -constraints while its dual has  $m$ -unknowns.
- (3) The primal has  $n$ -unknowns while its dual has  $n$ -constraints.
- (4) The  $n$ -coefficients of the objective function of primal ( $C_j$ ) become the  $n$ -constant terms ( $b_j$ ) of its dual.
- (5) The  $m$ -constant terms of the primal ( $b_j$ ) become the  $m$ -constant terms of the objective function ( $C_j$ ) of its dual.

- (6) The coefficients of the variables of the primal are transformed in their position in the dual. This means that the first column of the coefficients in the primal becomes the first row in the dual, the second column becomes the second row and so on.
- (7) The  $n$ -variables ( $X_j$ ) of the primal are replaced by the  $m$  new variables ( $Y_j$ ) of its dual. This change affects the system of restrictions as well as the objective function.
- (8) The sign of the inequalities in the set of restrictions of the primal ( $\leq$ ) is reversed in the set of restrictions in its dual ( $\geq$ ). In other words, if the inequalities in the primal are of the type  $\leq$ , then, they are of  $\geq$  type in the dual.
- (9) The sign of the inequalities restricting the variable ( $\geq X_j$ ) to non-negative values in the primal is equal to the inequality sign of the new variable ( $\geq Y_j$ ) of its dual.
- (10) For writing the dual of the given maximization problem, we should first ensure that all the constraint inequalities are of the  $\leq$  type and for writing the dual of the given minimization problem, the constraint inequalities should be of the  $\geq$  type. We can see the application of these considerations with the help of given examples.

Example 2.10: Write the dual of the following primal LP problem.

Maximize,

$$Z = 2X_1 + 3X_2$$

Subject to,

$$2X_1 + X_2 \leq 20$$

and

$$X_1 + 2X_2 \leq 20$$

$$X_1, X_2 \geq 0$$

Solution: The dual of the primal problem will be as under:

Minimize,

$$Z_1 = 20Y_1 + 20Y_2$$

Subject to,

$$2Y_1 + Y_2 \geq 2$$

and

$$Y_1 + 2Y_2 \geq 3$$

$$Y_1 \geq 0; Y_2 \geq 0$$

Example 2.11: Obtain the dual of the following LP problem.

Maximize,

$$Z_1 = 20X_1 + 30X_2 + 10X_3$$

Subject to,

$$X_1 - X_3 \leq 4$$

$$12X_1 + 8X_2 \leq 15$$

$$X_1 + X_2 + X_3 \geq 8$$

$$4X_1 + 3X_2 - X_3 = 10$$

$$X_1 \geq 0; X_2 \geq 0; X_3 \geq 0$$

Check Your Progress

9. When is an objective function minimized? When is it maximized?
10. What is meant by a feasible solution?
11. What is a feasible region?
12. What is an optimal solution?
13. What are non-degenerate and degenerate type basic feasible solutions?
14. Explain the simplex method.
15. How is a leaving element converted to unity in a simplex algorithm?
16. What is the role of the slack variable?
17. When is  $M$  method used?

Solution: The dual of this problem will be as under:

Minimize,

$$Z_p = 4Y_1 + 15Y_2 - 8Y_3 + 10Y_4 - 10Y_5$$

Subject to,

$$Y_1 + 12Y_2 - Y_3 + 4Y_4 - 4Y_5 \geq 20$$

$$0Y_1 + 8Y_2 - Y_3 + 3Y_4 - 3Y_5 \geq 30$$

$$-1Y_1 + 0Y_2 - Y_3 - Y_4 + Y_5 \geq 10$$

and

$$Y_1, Y_2, Y_3, Y_4, Y_5 \geq 0$$

Since the above form of dual contains 5 variables in the objective function which are only four constraints in the given primal problem. This situation can be tackled if we take  $Y_6$  as the difference between two non-negative variables  $Y_4$  and be  $Y_3$  (i.e.,  $Y_4 = Y_3 + Y_6$ ). Accordingly, the above dual can be written as under:

Minimize,

$$Z_p = 4Y_1 + 15Y_2 - 8Y_3 + 10Y_6$$

Subject to,

$$Y_1 + 12Y_2 - Y_3 + 4Y_6 \geq 20$$

$$8Y_2 - Y_3 + 3Y_6 \geq 30$$

$$-Y_1 - Y_3 - Y_6 \geq 10$$

and

$$Y_1, Y_2, Y_3 \geq 0 \text{ and } Y_6 \text{ unrestricted in sign.}$$

Note: Whenever a constraint in the given problem has an equality sign its corresponding dual variable shall be unrestricted in sign. Similarly if there is unrestricted variable on the given primal problem its corresponding constraint shall have the = sign.

Example 2.12: Write the dual of the following LP problem.

Maximize,

$$Z_x = 5X_1 + 7X_2 + 9X_3$$

Subject to,

$$2X_1 + 2X_2 + 5X_3 \leq 12$$

$$6X_1 - 2X_2 + 4X_3 \geq 15$$

and  $X_1, X_2 > 0$  and  $X_3$  unrestricted in sign.

Solution: The dual of the given problem will be written as under:

Minimize,

$$Z_v = 12Y_1 - 15Y_2$$

Subject to,

$$2Y_1 - 6Y_2 \geq 5$$

$$2Y_1 + 2Y_2 \geq 7$$

$$5Y_1 - 4Y_2 = 9$$

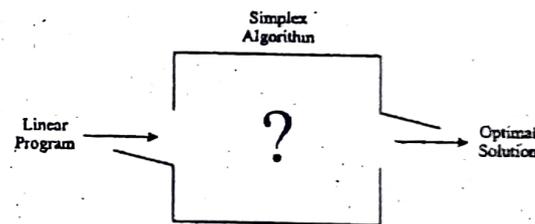
and

$$Y_1 \geq 0; Y_2 \geq 0$$

Since, in the given primal problem,  $X_3$  is unrestricted in sign, we have taken  $X_3 = (X_4 - X_5)$  and then using it the dual has been developed as above. It is instructive to note that the simplex method automatically identifies the dual basic solution. The optimal value of the objective function remains same as in the primal problem. Given an optimal solution of the primal problem, the  $i$ th dual variable acquires the coefficient of the  $i$ th slack variable in the optimal objective function equation as its optimal value. Thus, it is possible to identify the dual solution from the primal solution.

## 2.8 SENSITIVITY ANALYSIS

The optimal solution of a linear programming problem is formulated using various methods. You have learned the use and importance of dual variables to solve an LPP. Here, you will learn how sensitivity analysis helps to solve repeatedly the real problem in a little different form. Generally, these scenarios crop up as an end result of parameter changes due to the involvement of new advanced technologies and the accessibility of well-organized latest information for key (input) parameters or the 'what-if' questions. Thus, sensitivity analysis helps to produce optimal solution of simple perturbations for the key parameters. For optimal solutions, consider the simplex algorithm as a 'black box' which accepts the input key parameters to solve LPP as shown below:



Example 2.13: Illustrate sensitivity analysis using simplex method to solve the following LPP:

$$\text{Maximize } Z = 20x_1 + 10x_2$$

$$\text{Subject to, } x_1 + x_2 \leq 3$$

$$3x_1 + x_2 \leq 7$$

$$\text{And } x_1, x_2 \geq 0$$

Solution: Sensitivity analysis is done after making the initial and final tableau using the simplex method. Add slack variables to convert it into equation form.

$$\text{Maximize } Z = 20x_1 + 10x_2 + 0S_1 + 0S_2$$

$$\text{Subject to, } x_1 + x_2 + S_1 + 0S_2 = 3$$

$$3x_1 + x_2 + 0S_1 + S_2 = 7$$

$$\text{Where } x_1, x_2 \geq 0$$

To find basic feasible solution we put  $x_1 = 0$  and  $x_2 = 0$ . This gives  $Z = 0$ ,  $S_1 = 3$  and  $S_2 = 7$ . The initial table will be as follows:

NOTES

Initial table

		$C_j$	20	10	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	$\text{Min } \frac{X_B}{X_1}$
0	$S_1$	3	1	1	1	0	$3/1 = 3$
$\leftarrow 0$	$S_2$	7	3	1	0	1	$7/3 = 2.33$
	$Z_j$	0	0	0	0	0	
	$Z_j - C_j$		$-20 \uparrow$	$-10$	0	0	

Find  $\frac{X_B}{X_1}$  for each row and also find minimum for the second row. Here,  $Z_j - C_j$  is maximum negative ( $-20$ ). Hence,  $x_1$  enters the basis and  $S_2$  leaves the basis. It is shown with the help of arrows.

Key element is 3, key row is second row and key column is  $x_1$ . Now convert the key element into entering key by dividing each element of the key row by key element using the following formula:

New element = Old element

$$\left[ \frac{\text{Product of elements in the key row and key column}}{\text{Key element}} \right]$$

The following is the first iteration table:

		$C_j$	20	10	0	0	
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$	
$\leftarrow 0$	$S_1$	$2/3$	0	$2/3$	1	$-1/3$	$\frac{2}{3} / \frac{2}{3} = 1$
20	$X_1$	$7/3$	1	$1/3$	0	1	$\frac{7}{3} / \frac{1}{3} = 7$
	$Z_j$	$140/3$	20	$20/3$	0	20	
	$Z_j - C_j$		0	$-10/3 \uparrow$	0	-20	

Since  $Z_j - C_j$  has one value less than zero, i.e., negative value hence this is not yet the optima solution. Value  $-10/3$  is negative hence  $x_2$  enters the basis and  $S_1$  leaves the basis. Key row is upper row.

		$C_j$	20	10	0	0
$C_B$	$B$	$X_B$	$X_1$	$X_2$	$S_1$	$S_2$
10	$X_2$	1	0	1	$3/2$	$-1/2$
20	$X_1$	$4/3$	1	0	0	$4/3$
	$Z_j$	$110/3$	20	10	0	25
	$Z_j - C_j$		0	0	0	25

$Z_j - C_j \geq 0$  for all, hence optimal solution is reached, where  $x_1 = \frac{4}{3}$ ,  $x_2 = 1$ ,  
 $Z = \frac{80}{3} + 10 = \frac{110}{3}$

2.9 SUMMARY

- Linear Programming (LP) is a major innovation since World War II in the field of business decision-making, particularly under conditions of certainty.
- The word 'Linear' means that the relationships are represented by straight lines, i.e., the relationships are of the form  $y = a + bx$  and the word 'Programming' means taking decisions systematically.
- LP is generally used in solving maximization (sales or profit maximization) or minimization (cost minimization) problems subject to certain assumptions.
- LP is extensively used in solving resource allocation problems.
- The term linearity implies straight line or proportional relationships among the relevant variables.
- Process means the combination of particular inputs to produce a particular output.
- Criterion function is also known as objective function which states the determinants of the quantity either to be maximized or to be minimized.
- Feasible solutions are all those possible solutions which can be worked upon under given constraints. The region comprising of all feasible solutions is referred as feasible region.
- Optimum solution is the best of the feasible solutions.  $Z$  denotes the optimal value of the objective function for a given system.
- LP model is based on the assumptions of proportionality, additivity, certainty, continuity and finite choices.
- The applications of linear programming problems are based on linear programming matrix coefficients and data transmission prior to solving the simplex algorithm.
- The portfolio optimization template calculates the optimal capital of investments that gives the highest return for the least risk.
- The objective function, the set of constraints and the non-negative constraints together form a linear programming problem.
- Simple linear programming problem with two decision variables can be easily solved by graphical method.
- A set of values  $X_1, X_2, \dots, X_n$  which satisfies the constraints of the LPP is called its solution.
- Given a system of  $m$  linear equations with  $n$  variables ( $m < n$ ), any solution which is obtained by solving  $m$  variables keeping the remaining  $n - m$  variables

18. W  
19. W  
20. H  
se  
h

zero is called a basic solution. Such  $m$  variables are called basic variables and the remaining variables are called non-basic variables.

- A basic feasible solution is a basic solution which also satisfies all basic variables which are non-negative.
- If the value of the objective function  $Z$  can be increased or decreased indefinitely, such solutions are called unbounded solutions.
- Simplex method is an iterative procedure for solving LPP in a finite number of steps.
- For the solution of any LPP by simplex algorithm, the existence of an initial basic feasible solution is always assumed.
- In simplex algorithm, the  $M$  Method is used to deal with the situation where an infeasible starting basic solution is given.
- For every given linear programming problem there is another intimately related linear programming problem referred to as its dual.
- The duality theorem states that 'for every maximization (or minimization) problem in linear programming there is a unique similar problem of minimization (or maximization) involving the same data which describes the original problem'. The original problem is referred to as the 'primal'. The 'dual' of a dual problem is the primal. Thus the primal and dual problems are replicas of each other.
- The optimal solution of a linear programming problem is formulated using various methods.
- Sensitivity analysis helps to produce optimal solution of simple perturbations for the key parameters.

## 2.10 KEY TERMS

- **Linear programming:** A decision-making technique under a set of given constraints and is based on the assumption that the relationships amongst the variables representing different phenomena are linear
- **Decision variables:** Variables that form objective function and on which the cost or profit depends
- **Linearity:** It is the line straight or proportional relationships among the relevant variables. Linearity in economic theory is known as constant return
- **Process:** The combination of one or more inputs to produce a particular output.
- **Criterion function:** An objective function which states the determinants of the quantity to be either maximized or minimized
- **Constraints:** Limitations under which planning is decided. Restrictions imposed on decision variables
- **Feasible solution:** Any solution to a LPP which satisfies the non-negativity restrictions of the LPP
- **Feasible region:** The region comprising all feasible solutions

- **Optimal solution:** Any feasible solution which optimizes (minimizes or maximizes) the objective function of the LPP
- **Proportionality:** An assumption made in the objective function and constraint inequalities. In economic terminology this means that there are constant returns to scale
- **Certainty:** Assumption that includes prior knowledge of all the coefficients in the objective function, the coefficients of the constraints and the resource values. LP model operates only under conditions of certainty
- **Additivity:** An assumption which means that the total of all the activities is given by the sum total of each activity conducted separately
- **Continuity:** An assumption which means that the decision variables are continuous
- **Finite choices:** An assumption that implies that finite numbers of choices are available to a decision-maker and the decision variables do not assume negative values
- **Solution:** A set of values  $X_1, X_2, \dots, X_n$  which satisfies the constraints of the LPP
- **Basic solution:** In a given system of  $m$  linear equations with  $n$  variables ( $m < n$ ), any solution which is obtained by solving  $m$  variables keeping the remaining  $n - m$  variables zero is called a basic solution
- **Basic feasible solution:** A basic solution which also satisfies the condition in which all basic variables are non-negative
- **Canonical form:** It is irrespective of the objective function. All the constraints are expressed as equations and right hand side of each constraint and all variables are non-negative
- **Slack variables:** If the constraints of a general LPP be given as  $\sum a_{ij} X_j \leq b_i$  ( $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ ), then the non-negative variables  $S_i$  is introduced to convert the inequalities ' $\leq$ ' to the equalities are called slack variables
- **Surplus variables:** If the constraints of a general LPP be  $\sum a_{ij} X_j \geq b_i$  ( $i = 1, 2, \dots, m; j = 1, 2, \dots, n$ ), then non-negative variables  $S_j$  introduced to convert the inequalities ' $\geq$ ' to the equalities are called surplus variables

## 2.11 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Linear programming is a decision-making technique under a set of given constraints and is based on the assumption that the relationships amongst the variables representing different phenomena are linear.
2. Criterion function is objective function which states the determinants of the quantity, to be either maximized or minimized.
3. Linear programming finds application in agricultural and various industrial problems.
4. Constraints are limitations under which planning is decided, these are restrictions imposed on decision variables.

NOTES

5. Solution of a linear programming is a set of values  $X_1, X_2, \dots, X_n$ , satisfying the constraints of the LPP.
6. In a given system of  $m$  linear equations with  $n$  variables ( $m < n$ ), any solution which is obtained by solving  $m$  variables keeping the remaining  $n - m$  variables zero is called a basic solution.
7. In a given system of  $m$  linear equations with  $n$  variables ( $m < n$ ), where  $m$  variables are solved keeping remaining  $n - m$  variables zero,  $m$  variables are called basic variables and the remaining variables are called non-basic variables.
8. Basic feasible solution is a basic solution which satisfies the condition in which all basic variables are non-negative.
9. An objective function is maximized when it is a profit function. It is minimized when it is a cost function.
10. Feasible solution of a LPP is a solution that satisfies the non-negativity restrictions of the LPP.
11. Feasible region is the region comprising all feasible solutions.
12. Optimal solution of a LPP is a feasible solution which optimizes (minimizes or maximizes) the objective function of the LPP.
13. Non-degenerate and degenerate solutions are the basic feasible solutions. In a problem which has exactly  $m$  positive variables,  $X_i$  ( $i = 1, 2, \dots, m$ ), i.e., none of the basic variables is zero, then it is called non-degenerate type and if one or more basic variables are zero then such basic feasible solution is said to be degenerate type.
14. Simplex method is an iterative procedure for solving LPP in a finite number of steps. This method provides an algorithm which consists of moving from one vertex of the region of feasible solution to another in such a manner that the value of the objective function at the succeeding vertex is less or more as the case may be that at the previous vertex.
15. The leaving element is converted to unity by dividing the key equation by the key element and all other elements in its column to zero by using the formula:  

$$\text{New element} = \text{Old element} - \left[ \frac{\text{Product of elements in key row and key column}}{\text{Key element}} \right]$$
16. By introducing slack variable, the problem is converted into standard form.
17.  $M$  method is used to find the starting basic feasible solution each time it exists when an infeasible starting basic solution is given.
18. For every given linear programming problem there is another intimately related linear programming problem referred to as its dual. The duality theorem states that 'for every maximization (or minimization) problem in linear programming there is a unique similar problem of minimization (or maximization) involving the same data which describes the original problem'.
19. The original problem is referred to as the 'primal'. The 'dual' of a dual problem is the primal. Thus the primal and dual problems are replicas of each other.
20. Sensitivity analysis helps to produce optimal solution of simple perturbations for the key parameters.

2.12 QUESTIONS AND EXERCISES

Short-Answer Questions

1. What is meant by proportionality in linear programming?
2. What do you understand by certainty in linear programming?
3. What is meant by continuity in linear programming?
4. What are finite choices in the context of linear programming?
5. What are the basic constituents of an LP model?
6. What is the canonical form of a LPP?
7. What are characteristics of the canonical form?
8. What are slack variables? Where are they used? Explain in brief.
9. What do you understand by surplus variables?
10. What is the simplex method?
11. Does every LPP solution have an optimal solution? Explain.
12. What is the importance of the  $M$  method?
13. What is dual?
14. How will you transform a given primal problem into a dual problem?
15. What is the significance of sensitivity analysis?

Long-Answer Questions

1. A company manufactures 3 products  $A, B$  and  $C$ . The profits are: ₹ 3, ₹ 2 and ₹ 4 respectively. The company has two machines and given below is the required processing time in minutes for each machine on each product.

	Products		
Machines	A	B	C
I	4	3	5
II	2	2	4

- Machines I and II have 2000 and 2500 minutes respectively. The company must manufacture 100  $A$ 's 200  $B$ 's and 50  $C$ 's but no more than 150  $A$ 's. Find the number of units of each product to be manufactured by the company to maximize the profit. Formulate the above as a LP Model.
2. A company produces two types of leather belts  $A$  and  $B$ .  $A$  is of superior quality and  $B$  is of inferior quality. The respective profits are ₹ 10 and ₹ 5 per belt. The supply of raw material is sufficient for making 850 belts per day. For belt  $A$ , a special type of buckle is required and 500 are available per day. There are 700 buckles available for belt  $B$  per day. Belt  $A$  needs twice as much time as that required for belt  $B$  and the company can produce 500 belts if all of them were of the type  $A$ . Formulate a LP Model for the given problem.

- The standard weight of a special purpose brick is 5 kg and it contains two ingredients  $B_1$  and  $B_2$ , where  $B_1$  costs ₹ 5 per kg and  $B_2$  costs ₹ 8 per kg. Strength considerations dictate that the brick contains not more than 4 kg of  $B_1$  and a minimum of 2 kg of  $B_2$ , since the demand for the product is likely to be related to the price of the brick. Formulate the given problem as a LP Model.
- Egg contains 6 units of vitamin A per gram and 7 units of vitamin B per gram and 12 units of vitamin B per gram and costs 20 paise per gram. The daily minimum requirement of vitamin A and vitamin B are 100 units and 120 units respectively. Find the optimal product mix.
- A company desires to devote the excess capacity of the three machines lathe, shaping machine and milling machine to make three products A, B and C. The available time per month in these machinery are tabulated below:

Machine	Lathe	Shaping	Milling
Available Time/Month	200 hrs	100 hrs	180 hrs

The time taken to produce each unit of the products A, B and C on the machines is displayed in the table below.

	Lathe	Shaping	Milling
Product A hrs	6	2	4
Product B hrs	2	2	-
Product C hrs	3	-	3

The profit per product would be ₹ 20, ₹ 16 and ₹ 12 respectively on product A, B and C.

Formulate a LPP to find the optimum product-mix.

- Solve the following by graphical method:

(i) Max  $Z = X_1 - 3X_2$   
 Subject to,  $X_1 + X_2 \leq 300$   
 $X_1 - 2X_2 \leq 200$   
 $2X_1 + X_2 \leq 100$   
 $X_2 \leq 200$   
 $X_1, X_2 \geq 0$

(ii) Max  $Z = 5X + 8Y$   
 Subject to,  $3X + 2Y \leq 36$   
 $X + 2Y \leq 20$   
 $3X + 4Y \leq 42$   
 $X, Y \geq 0$

(iii) Max  $Z = X - 3Y$   
 Subject to,  $X + Y \leq 300$   
 $X - 2Y \leq 200$   
 $X + Y \leq 100$   
 $Y \geq 200$   
 and  $X, Y \geq 0$

- Solve graphically the following LPP:

Max  $Z = 20X_1 + 10X_2$   
 Subject to,  $X_1 + 2X_2 \leq 40$   
 $3X_1 + X_2 \geq 30$   
 $4X_1 + 3X_2 \geq 60$   
 and  $X_1, X_2 \geq 0$

- Using simplex method, find non-negative values of  $X_1, X_2$  and  $X_3$  when

(i) Max  $Z = X_1 + 4X_2 + 5X_3$   
 Subject to the constraints,  
 $3X_1 + 6X_2 + 3X_3 \leq 22$   
 $X_1 + 2X_2 + 3X_3 \leq 14$  and  
 $3X_1 + 2X_2 \leq 14$

(ii) Max  $Z = X_1 + X_2 + 3X_3$   
 Subject to,  $3X_1 + 2X_2 + X_3 \leq 2$   
 $2X_1 + X_2 + 2X_3 \leq 2$   
 $X_1, X_2, X_3 \geq 0$

(iii) Max  $Z = 10X_1 + 6X_2$   
 Subject to,  $X_1 + X_2 \leq 2$   
 $2X_1 + X_2 \leq 4$   
 $3X_1 + 8X_2 \leq 12$   
 $X_1, X_2 \geq 0$

(iv) Max  $Z = 30X_1 + 23X_2 + 29X_3$   
 Subject to the constraints,  
 $6X_1 + 5X_2 + 3X_3 \leq 52$   
 $6X_1 + 2X_2 + 5X_3 \leq 14$   
 $X_1, X_2, X_3 \geq 0$

(v) Max  $Z = X_1 + 2X_2 + X_3$   
 Subject to,  $2X_1 + X_2 - X_3 \geq -2$   
 $-2X_1 + X_2 - 5X_3 \leq 6$   
 $4X_1 + X_2 + X_3 \leq 6$   
 $X_1, X_2, X_3 \geq 0$

- Solve the following LPP applying M method:

Maximize,  $Z = 3X_1 + 4X_2$   
 Subject to,  $2X_1 + X_2 \leq 600$   
 $X_1 + X_2 \leq 225$   
 $5X_1 + 4X_2 \leq 1000$

NOTES

$$X_1 + 2X_2 \geq 150$$

$$X_1, X_2 \geq 0$$

10. Write the dual of the following LP problem:

$$\text{Maximize, } Z = 5X_1 + 6X_2$$

$$\text{Subject to, } X_1 + 2X_2 = 5$$

$$-X_1 + 5X_2 \geq 3$$

$$4X_1 + 7X_2 \leq 8$$

and  $X_1$  unrestricted in sign

$$X_2 \geq 0$$

11. Write the primal of the following dual:

$$\text{Minimize, } ZY = 10Y_1 + 8Y_2$$

$$\text{Subject to, } Y_1 + 2Y_2 \geq 5$$

$$2Y_1 - Y_2 \geq 12$$

$$Y_1 + 3Y_2 \geq 4$$

and  $Y_1 \geq 0; \geq Y_2$  unrestricted in sign.

---

### 2.13 FURTHER READING.

---

Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.

Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.

Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.

---

## MODULE III

---

# UNIT 3 TRANSPORTATION PROBLEM

Transportation

## Structure

- 3.0 Introduction
- 3.1 Unit Objectives
- 3.2 Basics of Transportation Problem
  - 3.2.1 Transportation Algorithm
- 3.3 Applications of Transportation Problem
  - 3.3.1 The Travelling Salesman Problem
  - 3.3.2 Mathematical Formulation
- 3.4 Solutions of Transportation Problem
  - 3.4.1 North West Corner Rule
  - 3.4.2 Least Cost or Matrix Minima Method
  - 3.4.3 Vogel's Approximation Method (VAM)
- 3.5 MODI Method
- 3.6 Special Cases
  - 3.6.1 Maximization Case in Transportation Problem
  - 3.6.2 Transshipment Problem
  - 3.6.3 Transshipment Model
- 3.7 Summary
- 3.8 Key Terms
- 3.9 Answers to 'Check Your Progress'
- 3.10 Questions and Exercises
- 3.11 Further Reading

NOTE

## 3.0 INTRODUCTION

This unit will discuss the transportation problem, which is a subclass of a Linear Programming Problem (LPP). Transportation problems deal with the objective of transporting various quantities of a single homogeneous commodity initially stored at various origins, to different destinations, in a way that keeps transportation cost at a minimum.

You will learn about applications of the transportation problem and solution and rules to solve such problems. The solution of any transportation problem is obtained in two stages, initial solution and optimal solution. There are three methods of obtaining an initial solution. These are: North West Corner Rule, Least Cost Method and Vogel's Approximation Method (VAM). VAM is preferred since the solution obtained this way is very close to the optimal solution. The optimal solution of any transportation problem is a feasible solution that minimizes the total cost. An optimal solution is the second stage of a solution obtained by improving the initial solution. The MODI method is used to obtain optimal solutions and optimality tests.

## 3.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Know about the transportation problem
- Explain the three methods of finding an initial solution

NOTES

- Use the MODI method for finding optimal solution
- Understand special cases in transportation
- Explain restricted transportation problem

### 3.2 BASICS OF TRANSPORTATION PROBLEM

The transportation problem is one of the subclasses of LPP (Linear Programming Problem) in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum. To achieve this objective we must know the amount and location of available supplies and the quantities demanded. In addition we must know the costs that result from transporting one unit of commodity from various origins to various destinations.

#### Elementary Transportation Problem

Consider a transportation problem with  $m$  origins (rows) and  $n$  destinations (columns). Let  $C_{ij}$  be the cost of transporting one unit of the product from the  $i$ th origin to  $j$ th destination,  $a_i$  the quantity of commodity available at origin  $i$ ,  $b_j$  the quantity of commodity needed at destination  $j$ .  $X_{ij}$  is the quantity transported from  $i$ th origin to  $j$ th destination. This transportation problem can be stated in the following tabular form.

		Destinations					Capacity
		1	2	3	...	$n$	
Origins	1	$C_{11}$ $X_{11}$	$C_{12}$ $X_{12}$	$C_{13}$ $X_{13}$	...	$C_{1n}$ $X_{1n}$	$a_1$
	2	$C_{21}$ $X_{21}$	$C_{22}$ $X_{22}$	$C_{23}$ $X_{23}$	...	$C_{2n}$ $X_{2n}$	$a_2$
	3	$C_{31}$ $X_{31}$	$C_{32}$ $X_{32}$	$C_{33}$ $X_{33}$	...	$C_{3n}$ $X_{3n}$	$a_3$
	$m$	$C_{m1}$ $X_{m1}$	$C_{m2}$ $X_{m2}$	$C_{m3}$ $X_{m3}$	...	$C_{mn}$ $X_{mn}$	$a_m$
Demand		$b_1$	$b_2$	$b_3$	...	$b_n$	$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$

The linear programming model representing the transportation problem is given by,

$$\text{Minimize } Z = \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^n X_{ij} = a_i \quad i = 1, 2, \dots, m \text{ (Row Sum)}$$

$$\sum_{i=1}^m X_{ij} = b_j \quad j = 1, 2, \dots, n \text{ (Column Sum)}$$

$$X_{ij} \geq 0 \quad \text{For all } i \text{ and } j$$

The given transportation problem is said to be balanced if,

$$\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$$

i.e., the total supply is equal to the total demand.

#### Definitions

**Feasible Solution:** Any set of non-negative allocations ( $X_{ij} \geq 0$ ) which satisfies the row and column sum (rim requirement) is called a feasible solution.

**Basic Feasible Solution:** A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to  $m + n - 1$ , where  $m$  is the number of rows and  $n$  the number of columns in a transportation table.

**Non-Degenerate Basic Feasible Solution:** Any feasible solution to a transportation problem containing  $m$  origins and  $n$  destinations is said to be non-degenerate, if it contains  $m + n - 1$  occupied cells and each allocation is in independent positions.

The allocations are said to be in independent positions if it is impossible to form a closed path. Closed path means by allowing horizontal and vertical lines and when all the corner cells are occupied.

The allocations in the following tables are not in independent positions.

The allocations in the following tables are in independent positions.

**Degenerate Basic Feasible Solution:** If a basic feasible solution contains less than  $m + n - 1$  non-negative allocations it is said to be degenerate.

#### 3.2.1 Transportation Algorithm

This algorithm can be used for minimizing the transportation cost for goods from  $O$  origins to  $D$  destinations and there may be  $O \cdot D$  number of direct routes from  $O$  origins to  $D$  destinations. Problem is balanced when sum of supplies at  $O$  sources is equal to sum of demands at  $D$  destinations. If it is not so, then this problem is not

balanced. There may be two such situations. Supply may be lesser than demand and in that case it is balanced by adding dummy supply node. If demand is lesser than supply then dummy demand node is added to make it a balanced problem. Thus, before starting to use this algorithm, problem should be made balanced, if it is not balanced. The algorithm for balanced problem is being described below.

Data is presented in tabular form. As a convention, origins are put on left side of the table with quantity to be supplied listed towards right side and demands are put on top with quantity of demand towards the bottom side. Unit cost of transportation is put at the top of every cell within a small box. Zero unit cost shows unshipped unit column in case supply is in excess of the demand. Similarly, a unit cost either penalty or zero shows shortage row in supplies that are lesser than demand.

The algorithm has two phases. In phase I this makes allocation for supplies on demands by making utilizing an approach of minimum unit cost for generating a feasible solution. This feasible solution may not be optimal. Optimization is done in second phase and in this phase checking is done for optimality conditions and improvement is done for reducing the cost if optimality conditions are not satisfied. This second phase adopts iterative steps and stops only when optimality conditions are satisfied. Once done, no further steps are required.

### Basic and Non-Basic Cells

Basic cells are those that indicate positive values and non-basic cells have zero value for flow. According to transportation problem number of basic cells will be exactly  $m + n - 1$ .

Algorithm follows as below:

#### Step 0

**Initialization:** Before starting to solve the problem, it should be balanced. If not then make it balanced by 'unshipped supply' column in case demand is less than supply or by adding 'shortage' row in case supply is less than the supply. Put zero for unit costs in the column for unshipped supply. Put either penalty costs or zero in a row that shows shortage.

#### Phase I: To Find Initial 'Feasible Solution'

**Step 1:** Locate cell with minimum cost having positive supply as well as demand, then make allocation in that cell having residuals as minimum.

**Step 2:** Reduce residual supply/demand as per allocations made above (step 1). Do this till all demands are met and then proceed to Phase II. If all demands are not met go to step 1.

#### Phase II: Optimal Solution

Carry out check for Optimality.

**Step 3:** For rows and columns find dual values,  $u_i$  for rows and  $v_j$  for columns. For this, set  $u_1$  (first dual value) to 0 followed by solution of triangular dual equations one by one. These dual equations are to be applied for basic cells only as below as,

$$C_{ij} = u_i + v_j$$

Where,  $C_{ij}$  denotes unit cost, as given for that cell and either  $v_j$  or  $u_i$  is already known. Here,  $v_j$  denotes dual value of column  $j$  and  $u_i$  denotes that for row  $i$ . These are taken as already known. Other dual value is computed from equation,

$$C_{ij} = u_i + v_j$$

Thus, dual values of every cell can be computed by setting the first to zero and appropriate order is used for dual equations for basic cells.

**Step 4:** Optimality conditions are expressed as reduced costs in case of all non-basic cells as given below:

$$\Delta_{ij} = C_{ij} - (u_i + v_j)$$

Reduced cost ( $\Delta_{ij}$ ) in case of non-basic cells represent net change in unit cost resulting due to movement of cell  $ij$  to solve and adjusting around a cycle that has been created in this way for basic cells to find current solution. This is shown in step 5. Hence if one  $\Delta_{ij}$  is positive, then by use of this cell total transportation will increase, but when  $\Delta_{ij}$  is negative for a cell, this will cause reduction in total transportation. If all reduced costs ( $\Delta_{ij}$ ) are positive, i.e.,  $\Delta_{ij} > 0$ , optimality conditions are satisfied and no further improvement is possible and algorithm terminates at this point. But if at least one  $\Delta_{ij}$  is negative, optimality conditions are not satisfied and there is possibility of reduction in costs. If optimality conditions are not satisfied algorithm continues as given in Step 5.

#### Adjustment for Reducing Cost

**Step 5:** Select a cell 'ij' that is most negative for  $\Delta_{ij}$ . This becomes entering variable. Put (+) sign for identifying it. To maintain constraints of balance in supply and demand, locate basic cells for  $i$ th row as well as  $j$ th column that compensates for increase in value in cell 'ij'. Put negative (-) sign in these cells. This process should be continued to get one cycle in which (+) and (-) are marked. Such a cycle will unique and if 'dead ends' are encountered in such a process, make a back track and one amongst other alternatives are tried. A cycle has rows/column having non-basic cells for holding compensating (+) or (-) sign. This may require trial and error for finding it.

**Step 6:** After determining every basic cell within this cycle, adjustment is obtained as minimum value in basic cells that are negative. This is known as adjustment amount and let it be called 'aa'. Add this to every cell value that is positive and marked with (+) sign, and subsequently deduct this from cells having negative (-) sign and then drop those from the basis that becomes zero. In case two or more cells become zero from such adjustment, drop only one of these that have greatest  $C_{ij}$  value. This is necessary for maintaining basic cells having  $m + n - 1$  number for computing dual values. Reduction in cost that is associated with such a change is found as product of reduced cost  $\Delta_{ij}$  for incoming cell multiplied by cell value previously held by outgoing cell.

After finding this new solution, move to the third step 3 for checking conditions of optimality. If optimality condition is satisfied, algorithm terminates.

### 3.3 APPLICATIONS OF TRANSPORTATION PROBLEM

NOTES

#### 3.3.1 The Travelling Salesman Problem

Assume that a salesman has to visit  $n$  cities. He wishes to start from a particular city, visits each city once and then returns to his starting point. His objective is to select the sequence in which the cities are visited in such a way that his total travelling time is minimized.

To visit 2 cities A and B, there is no choice. To visit 3 cities we have  $2!$  possible routes. For 4 cities we have  $3!$  possible routes. In general to visit  $n$  cities there are  $(n-1)!$  possible routes.

#### 3.3.2 Mathematical Formulation

Let  $C_{ij}$  be the distance or time or cost of going from city  $i$  to city  $j$ . The decision variable  $X_{ij}$  be 1 if the salesman travels from city  $i$  to city  $j$  and otherwise 0.

The objective is to minimize the travelling time.

$$Z = \sum_{i=1}^n \sum_{j=1}^n C_{ij} X_{ij}$$

Subject to the constraints,

$$\sum_{j=1}^n X_{ij} = 1, i = 2, \dots, n$$

$$\sum_{i=1}^n X_{ij} = 1, j = 2, \dots, n$$

Subject to the additional constraint that  $X_{ij}$  is so chosen that no city is visited twice before all the cities are completely visited.

In particular, going directly from  $i$  to  $i$  is not permitted. Which means  $C_{ii} = \infty$ , when  $i = j$ .

In travelling salesman problem we cannot choose the element along the diagonal and this can be avoided by filling the diagonal with infinitely large elements.

The travelling salesman problem is very similar to the assignment problem except that in the former case, there is an additional restriction that  $X_{ij}$  is so chosen that no city is visited twice before the tour of all the cities is completed.

Treat the problem as an assignment problem and solve it using the same procedures. If the optimal solution of the assignment problem satisfies the additional constraint, then it is also an optimal solution of the given travelling salesman problem. If the solution to the assignment problem does not satisfy the additional restriction then after solving the problem by assignment technique we use the method of enumeration.

**Example 3.1:** A travelling salesman has to visit 5 cities. He wishes to start from a particular city, visit each city once and then return to his starting point. Cost of going from one city to another is shown below. You are required to find the least cost route.

		To City				
		A	B	C	D	E
From City	A	$\infty$	4	10	14	2
	B	12	$\infty$	6	10	4
	C	16	14	$\infty$	8	14
	D	24	8	12	$\infty$	10
	E	2	6	4	16	$\infty$

**Solution:** First we solve this problem as an assignment problem. Subtract the minimum element in each row from all the elements in its row.

	A	B	C	D	E
A	$\infty$	2	8	12	0
B	8	$\infty$	2	6	0
C	8	6	$\infty$	0	6
D	16	0	4	$\infty$	2
E	0	4	2	14	$\infty$

Subtract the minimum element in each column from all the elements in its column.

	A	B	C	D	E
A	$\infty$	2	8	12	0
B	8	$\infty$	2	6	0
C	8	6	$\infty$	0	6
D	16	0	4	$\infty$	2
E	0	4	2	14	$\infty$

We have the first modified matrix. Draw minimum number of lines to cover all zeros.

	A	B	C	D	E
A	$\infty$	2	8	12	0
B	8	$\infty$	2	6	0
C	8	6	$\infty$	0	6
D	16	0	4	$\infty$	2
E	0	4	2	14	$\infty$

Here  $N = 4 < n = 5$ . Subtract the smallest uncovered element from all the uncovered elements and add to the element which is in the point of intersection of lines. Hence, we get the second modified matrix.

	A	B	C	D	E
A	$\infty$	0	6	12	0
B	8	$\infty$	0	6	0
C	8	4	$\infty$	0	6
D	18	0	4	$\infty$	4
E	0	2	0	14	$\infty$

$N = 5 = n = 5 =$  Order of matrix. We make assignment.

	A	B	C	D	E	
A	$\infty$	X	6	12	0	A $\rightarrow$ E
B	8	$\infty$	0	6	X	B $\rightarrow$ C
C	8	4	$\infty$	0	6	C $\rightarrow$ D
D	18	0	4	$\infty$	4	D $\rightarrow$ B
E	0	2	X	14	$\infty$	E $\rightarrow$ A
						A $\rightarrow$ E $\rightarrow$ A

The salesman should go from A to E and then come back to A without covering B, C, D which is contradicting the fact that no city is visited twice before all the cities are visited.

Hence, we obtain the next best solution by bringing the next minimum non-zero element namely 4.

$$\begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{ccccc}
 & A & B & C & D & E \\
 A & \infty & 3 & \textcircled{1} & 6 & 12 & 0 \\
 B & 8 & \infty & \textcircled{1} & 6 & \infty & 0 \\
 C & 8 & 4 & \infty & \textcircled{1} & 6 & 0 \\
 D & 18 & \infty & 4 & \infty & 0 & \textcircled{1} \\
 E & \textcircled{1} & 2 & \infty & 14 & \infty & 0
 \end{array}$$

$A \rightarrow B, B \rightarrow C, C \rightarrow D, D \rightarrow E, E \rightarrow A$

Since all the cities have been visited and no city is visited twice before completing the tour of all the cities, we have an optimal solution to the travelling salesman.

The least cost route is  $A \rightarrow B \rightarrow C \rightarrow D \rightarrow E \rightarrow A$ .

Total Cost =  $4 + 6 + 8 + 10 + 2 = ₹30$ .

Example 3.2: A machine operator processes five types of items on his machine each week and must choose a sequence for them. The set-up cost per change depends on the items presently on the machine and the set-up to be made according to the following table.

$$\begin{array}{c}
 \text{To Item} \\
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{ccccc}
 & A & B & C & D & E \\
 A & \infty & 4 & 7 & 3 & 4 \\
 B & 4 & \infty & 6 & 3 & 4 \\
 C & 7 & 6 & \infty & 7 & 5 \\
 D & 3 & 3 & 7 & \infty & 7 \\
 E & 4 & 4 & 5 & 7 & \infty
 \end{array}$$

If he processes each type of item once and only once in each week, how should he sequence the items on his machine in order to minimize the total set-up cost?

Solution: Reduce the cost matrix and make assignments in rows and columns having single row.

Modify the matrix by subtracting the least element from all the elements in its row and also in its column.

$$\begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{ccccc}
 & A & B & C & D & E \\
 A & \infty & 1 & 4 & 0 & 1 \\
 B & 1 & \infty & 3 & 0 & 1 \\
 C & 2 & 1 & \infty & 2 & 0 \\
 D & 0 & 0 & 4 & \infty & 4 \\
 E & 0 & 0 & 1 & 3 & \infty
 \end{array}$$

$$\begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{ccccc}
 & A & B & C & D & E \\
 A & \infty & 1 & 4 & 0 & 1 \\
 B & 1 & \infty & 3 & 0 & 1 \\
 C & 2 & 1 & \infty & 2 & 0 \\
 D & 0 & 0 & 4 & \infty & 4 \\
 E & 0 & 0 & 1 & 3 & \infty
 \end{array}$$

Here,  $N = 4 < n = 5$ , i.e.,  $N < n$ .

Subtract the smallest uncovered element from all the uncovered elements and add to the element which is at the point of intersection of lines and get the reduced second modified matrix.

$$\begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{ccccc}
 & A & B & C & D & E \\
 A & \infty & 0 & 3 & 0 & 1 \\
 B & 0 & \infty & 2 & 0 & 1 \\
 C & 1 & 0 & \infty & 1 & 0 \\
 D & 0 & 0 & 4 & \infty & 4 \\
 E & 0 & 0 & 1 & 4 & \infty
 \end{array}$$

Here,  $N = 5 = n = 5 =$  Order of matrix. We make the assignment.

**Assignment**

We make a solution by considering the next smallest non-zero element by considering 1.

$$\begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{c}
 A \\
 B \\
 C \\
 D \\
 E
 \end{array}
 \begin{array}{ccccc}
 & A & B & C & D & E \\
 A & \infty & \infty & 3 & \infty & \textcircled{1} \\
 B & \infty & \infty & 2 & \textcircled{1} & 1 \\
 C & 1 & \textcircled{1} & \infty & 1 & \infty \\
 D & \textcircled{1} & \infty & 4 & \infty & 5 \\
 E & \infty & \infty & \textcircled{1} & 4 & \infty
 \end{array}
 \begin{array}{l}
 A \rightarrow E \\
 B \rightarrow D \\
 C \rightarrow B \\
 D \rightarrow A \\
 E \rightarrow C
 \end{array}$$

$A \rightarrow E, E \rightarrow C, C \rightarrow B, B \rightarrow D, D \rightarrow A$

We get the solution  $A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$ .

This schedule provides the required solution in which each item is not processed once in a week.

i.e.,  $A \rightarrow E \rightarrow C \rightarrow B \rightarrow D \rightarrow A$ .

The total set-up cost comes to ₹21.

**3.4 SOLUTIONS OF TRANSPORTATION PROBLEM**

Optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost.

The solution of a transportation problem (TP) can be obtained in two stages, namely initial solution and optimum solution.

Initial solution can be obtained by using any one of the three methods, viz.

- (i) North West Corner Rule (NWCR)
- (ii) Least Cost Method or Matrix Minima Method
- (iii) Vogel's Approximation Method (VAM)

VAM is preferred over the other two methods, since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.

The cells in the transportation table can be classified as occupied cells and unoccupied cells. The allocated cells in the transportation table is called *occupied cells* and *empty cells* in a transportation table is called *unoccupied cells*.

The improved solution of the initial basic feasible solution is called optimal solution which is the second stage of solution, that can be obtained by MODI (Modified Distribution Method).

NOTES

3.4.1 North West Corner Rule

**Step 1:** Starting with the cell at the upper left corner (North West) of the transportation matrix we allocate as much as possible so that either the capacity of the first row is exhausted or the destination requirement of the first column is satisfied, i.e.,  $X_{11} = \min(a_1, b_1)$ .

**Step 2:** If  $b_1 > a_1$ , we move down vertically to the second row and make the second allocation of magnitude  $X_{21} = \min(a_2, b_1 - X_{11})$  in the cell (2, 1).

If  $b_1 < a_1$ , move right horizontally to the second column and make the second allocation of magnitude  $X_{12} = \min(a_1, X_{11} - b_1)$  in the cell (1, 2).

If  $b_1 = a_1$ , there is a tie for the second allocation. We make the second allocations of magnitude,

$X_{12} = \min(a_1 - a_1, b_1) = 0$  in the cell (1, 2)

or,  $X_{21} = \min(a_2, b_1 - b_1) = 0$  in the cell (2, 1)

**Step 3:** Repeat steps 1 and 2 moving down towards the lower right corner of the transportation table until all the rim requirements are satisfied.

**Example 3.3:** Obtain the initial basic feasible solution of a transportation problem whose cost and rim requirement table is as follows:

Origin\Destination	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	Supply
O <sub>1</sub>	2	7	4	5
O <sub>2</sub>	3	3	1	8
O <sub>3</sub>	5	4	7	7
O <sub>4</sub>	1	6	2	14
Demand	7	9	18	34

**Solution:** Since  $\sum a_i = 34 = \sum b_j$ , there exists a feasible solution to the transportation problem. We obtain initial feasible solution as follows.

The first allocation is made in the cell (1, 1) the magnitude being,

$X_{11} = \min(5, 7) = 5$

The second allocation is made in the cell (2, 1) and the magnitude of the allocation is given by  $X_{21} = \min(8, 7 - 5) = 2$ .

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	Supply
O <sub>1</sub>	5		4	5 0
O <sub>2</sub>	2	3	1	8 6 0
O <sub>3</sub>		4	7	7 4 0
O <sub>4</sub>	1	6	2	14 0
Demand	7	9	18	34
	0	0	0	

The third allocation is made in the cell (2, 2) the magnitude  $X_{22} = \min(8 - 2, 9) = 6$ .

The magnitude of the fourth allocation is made in the cell (3, 2) given by  $X_{32} = \min(7, 9 - 6) = 3$ .

The fifth allocation is made in the cell (3, 3) with magnitude  $X_{33} = \min(7 - 3, 14) = 4$ .

The final allocation is made in the cell (4, 3) with magnitude  $X_{43} = \min(14, 18 - 4) = 14$ .

Hence, we get the initial basic feasible solution to the given TP and is given by,

$X_{11} = 5; X_{21} = 2; X_{22} = 6; X_{32} = 3; X_{33} = 4; X_{43} = 14$

Total Cost =  $2 \times 5 + 3 \times 2 + 3 \times 6 + 3 \times 4 + 4 \times 7 + 2 \times 14$   
 $= 10 + 6 + 18 + 12 + 28 + 28 = ₹102$

**Example 3.4:** Determine an initial basic feasible solution to the following transportation problem using North West Corner Rule.

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply
O <sub>1</sub>	6	4	1	5	14
O <sub>2</sub>	8	9	2	7	16
O <sub>3</sub>	4	3	6	2	5
Required	6	10	15	4	35

**Solution:** The problem is a balanced TP as the total supply is equal to the total demand. Using the steps we find the initial basic feasible solution as given in the following table.

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply
O <sub>1</sub>	6	4	1	5	14 8 0
O <sub>2</sub>	8	9	2	7	16 4 0
O <sub>3</sub>	4	3	6	2	5
Demand	6	10	15	4	35

The solution is given by,

$X_{11} = 6; X_{12} = 8; X_{22} = 2; X_{23} = 14; X_{33} = 1; X_{34} = 4$

Total Cost =  $6 \times 6 + 4 \times 8 + 2 \times 9 + 2 \times 14 + 6 \times 1 + 2 \times 4$   
 $= 36 + 32 + 18 + 28 + 6 + 8 = ₹128$

3.4.2 Least Cost or Matrix Minima Method

**Step 1:** Determine the smallest cost in the cost matrix of the transportation table. Let it be  $C_{ij}$ . Allocate  $X_{ij} = \min(a_i, b_j)$  in the cell (i, j).

**Step 2:** If  $X_{ij} = a_i$ , cross off the *i*th row of the transportation table and decrease  $b_j$  by  $a_i$ . Then go to step 3.

If  $X_{ij} = b_j$ , cross off the *j*th column of the transportation table and decrease  $a_i$  by  $b_j$ . Go to step 3.

If  $X_{ij} = a_i = b_j$ , cross off either the *i*th row or the *j*th column but not both.

**Step 3:** Repeat Steps 1 and 2 for the resulting reduced transportation table until all the rim requirements are satisfied. Whenever the minimum cost is not unique, make an arbitrary choice among the minima.

**Example 3.5:** Obtain an initial feasible solution to the following TP using Matrix Minima Method.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	1	2	3	4	6
$O_2$	4	3	2	0	8
$O_3$	0	2	2	1	10
Demand	4	6	8	6	24

Solution: Since  $\sum a_i = \sum b_j = 24$ , there exists a feasible solution to the TP using the steps in the least cost method, the first allocation is made in the cell (3, 1) the magnitude being  $X_{31} = 4$ . This satisfies the demand at the destination  $D_1$ , and we delete this column from the table as it is exhausted.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	1	2	3	4	6
$O_2$	4	3	2	0	8
$O_3$	0	2	2	1	10
Demand	0	6	8	6	24

The second allocation is made in the cell (2, 4) with magnitude  $X_{24} = \min(6, 8) = 6$ . Since it satisfies the demand at the destination  $D_4$ , it is deleted from the table. From the reduced table, the third allocation is made in the cell (3, 3) with magnitude  $X_{33} = \min(8, 6) = 6$ . The next allocation is made in the cell (2, 3) with magnitude  $X_{23} = \min(2, 2) = 2$ . Finally, the allocation is made in the cell (1, 2) with magnitude  $X_{12} = \min(6, 6) = 6$ . Now, all the requirements have been satisfied and hence, the initial feasible solution is obtained.

The solution is given by,

$$X_{12} = 6; X_{23} = 2; X_{24} = 6; X_{31} = 4; X_{33} = 6$$

Since the total number of occupied cells = 5 < m + n - 1

We get a degenerate solution.

$$\begin{aligned} \text{Total Cost} &= 6 \times 2 + 2 \times 2 + 6 \times 0 + 4 \times 0 + 6 \times 2 \\ &= 12 + 4 + 12 = ₹28. \end{aligned}$$

**Example 3.6:** Determine an initial basic feasible solution for the following TP, using the Least Cost Method.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	6	4	1	5	14
$O_2$	8	9	2	7	16
$O_3$	4	3	6	2	5
Demand	6	10	15	4	35

Solution: Since  $\sum a_i = \sum b_j$ , there exists a basic feasible solution. Using the steps in least cost method we make the first allocation to the cell (1, 3) with magnitude  $X_{13} = \min(14, 15) = 14$  (as it is the cell having the least cost).

This allocation exhausts the first row supply. Hence, the first row is deleted. From the reduced table, the next allocation is made in the next least cost cell (2, 3)

which is chosen arbitrarily with magnitude  $X_{23} = \min(1, 16) = 1$ . This exhausts the 3rd column destination.

From the reduced table the next least cost cell is (3, 4) for which allocation is made with magnitude  $\min(4, 5) = 4$ . This exhausts the destination  $D_4$  requirement. Delete this fourth column from the table. The next allocation is made in the cell (3, 2) with magnitude  $X_{32} = \min(1, 10) = 1$  which exhausts the 3rd origin capacity. Hence, the 3rd row is exhausted. From the reduced table the next allocation is given to the cell (2, 1) with magnitude  $X_{21} = \min(6, 15) = 6$ . This exhausts the first column requirement. Hence, it is deleted from the table.

Finally, the allocation is made to the cell (2, 2) with magnitude  $X_{22} = \min(9, 9) = 9$  which satisfies the rim requirement. These allocation are shown in the transportation table as follows:

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	16	4	1	5	14
$O_2$	8	9	2	7	16
$O_3$	4	3	6	2	5
Demand	6	10	15	4	

(I Allocation)

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_2$	8	9	2	7	16
$O_3$	4	3	6	2	5
Demand	6	10	15	4	

(II Allocation)

	$D_1$	$D_2$	$D_3$	Supply
$O_2$	8	9	7	15
$O_3$	4	3	2	5
Demand	6	10	4	

(III Allocation)

	$D_1$	$D_2$	Supply
$O_2$	8	9	15
$O_3$	4	3	5
Demand	6	9	

(IV Allocation)

	$D_1$	$D_2$	Supply
$O_2$	8	9	15
Demand	6	9	

(V, VI Allocation)

The following table gives the initial basic feasible solution.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	6	4	1	5	14
$O_2$	8	9	2	7	16
$O_3$	4	3	6	2	5
Demand	6	10	15	4	

The solution is given by,

$$X_{13} = 14; X_{21} = 6; X_{22} = 9; X_{23} = 1; X_{32} = 1; X_{34} = 4$$

Transportation Cost,

$$\begin{aligned} &= 14 \times 1 + 6 \times 8 + 9 \times 9 + 1 \times 2 + 3 \times 1 + 4 \times 2 \\ &= 14 + 48 + 81 + 2 + 3 + 8 = ₹156 \end{aligned}$$

NOTES

### 3.4.3 Vogel's Approximation Method (VAM)

The steps involved in this method for finding the initial solution are as follows.

**Step 1:** Find the penalty cost, namely the difference between the smallest and next smallest costs in each row and column.

**Step 2:** Among the penalties as found in Step (1) choose the maximum penalty. If this maximum penalty is more than one (i.e., if there is a tie) choose any one arbitrarily.

**Step 3:** In the selected row or column as by Step (2) find out the cell having the least cost. Allocate to this cell as much as possible depending on the capacity and requirements.

**Step 4:** Delete the row or column which is fully exhausted. Again, compute the column and row penalties for the reduced transportation table and then go to Step (2). Repeat the procedure until all the rim requirements are satisfied.

**Note:** If the column is exhausted, then there is a change in row penalty and vice versa.

**Example 3.7:** Find the initial basic feasible solution for the following transportation problem using VAM.

		Destination				Supply
		D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	
Origin	O <sub>1</sub>	11	13	17	14	250
	O <sub>2</sub>	16	18	14	10	300
	O <sub>3</sub>	21	24	13	10	400
	Demand	200	225	275	250	950

**Solution:** Since  $\sum a_i = \sum b_j = 950$ , the problem is balanced and there exists a feasible solution to the problem.

First, we find the row and column penalty  $P_i$  as the difference between the least and the next least cost. The maximum penalty is 5. Choose the first column arbitrarily. In this column, choose the cell having the least cost name (1, 1). Allocate to this cell with minimum magnitude (i.e.,  $\text{Min}(250, 200) = 200$ .) This exhausts the first column. Delete this column. Since a column is deleted, then there is a change in row penalty  $P_i$  and column penalty  $P_j$  remains the same. Continuing in this manner, we get the remaining allocations as given in the following table below.

I Allocation

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply	P <sub>i</sub>
O <sub>1</sub>	11 200	13	17	14	50 250	2
O <sub>2</sub>	16	18	14	10	300	4
O <sub>3</sub>	21	24	13	10	400	3
Demand	200	225	275	250		
P <sub>i</sub>	5↑	5	1	0		

II Allocation

	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply	P <sub>ii</sub>
O <sub>1</sub>	13 50	17	14	50 250	1
O <sub>2</sub>	18	14	10	300	4
O <sub>3</sub>	24	13	10	400	3
Demand	225	275	250		
P <sub>ii</sub>	5↑	1	0		

III Allocation

	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply	P <sub>iii</sub>
O <sub>2</sub>	18 175	14	10	300 125	4
O <sub>3</sub>	24	13	10	400	3
Demand	175	275	250		
P <sub>iii</sub>	6↑	1	0		

IV Allocation

	D <sub>3</sub>	D <sub>4</sub>	Supply	P <sub>iv</sub>
O <sub>2</sub>	14 125	10	125 0	4 ←
O <sub>3</sub>	13	10	400	3
Demand	275	250	125	
P <sub>iv</sub>	1	0		

V Allocation

	D <sub>3</sub>	D <sub>4</sub>	Supply	P <sub>v</sub>
O <sub>3</sub>	13 125	10	400 125	3
Demand	275	125		
P <sub>v</sub>	13↑	10		

VI Allocation

	D <sub>4</sub>	Supply	P <sub>vi</sub>
O <sub>3</sub>	10 125	125 0	10 ←
Demand	125		
P <sub>vi</sub>	10		

Finally, we arrive at the initial basic feasible solution which is shown in the following table.

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply
O <sub>1</sub>	11 200	13 50	17	14	250
O <sub>2</sub>	16	18 175	14	10 125	300
O <sub>3</sub>	21	24	13 275	10 125	400
Demand	200	225	275	250	

There are 6 positive independent allocations which equals to  $m + n - 1 = 3 + 4 - 1$ . This ensures that the solution is a non-degenerate basic feasible solution.

∴ Transportation Cost,

$$= 11 \times 200 + 13 \times 50 + 18 \times 175 + 10 \times 125 + 13 \times 275 + 10 \times 125 = ₹12,075.$$

**Example 3.8:** Find the initial solution to the following TP using VAM.

		Destination				Supply
		D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	
Factory	F <sub>1</sub>	3	3	4	1	100
	F <sub>2</sub>	4	2	4	2	125
	F <sub>3</sub>	1	5	3	2	75
	Demand	120	80	75	25	300

Solution: Since  $\sum a_i = \sum b_j$ , the problem is a balance TP. Hence, there exists a feasible solution.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply	$P_I$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$	$P_{VI}$
$F_1$	3 (45)	3	4 (30)	1 (25)	100	2	2	0	1	4	4
$F_2$	4	2 (80)	2 (45)	2	125	0	0	2	0	4	←
$F_3$	1 (25)	5	3	2	75	1	←	←	←	←	←
Demand	120	80	75	25							
$P_I$	2↑	1	1	1							
$P_{II}$	1	1	0	1							
$P_{III}$	1	1	0								
$P_{IV}$	1		0								
$P_V$			0								
$P_{VI}$			4↑								

Finally, we have the initial basic feasible solution as given in the following table.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$F_1$	3 (45)		4 (30)	1 (25)	100
$F_2$	4	2 (80)	2 (45)	2	125
$F_3$	1 (25)		3	2	75
Demand	120	80	75	25	

There are 6 independent non-negative allocations equal to  $m + n - 1 = 3 + 4 - 1 = 6$ .

This ensures that the solution is non-degenerate basic feasible.

∴ Transportation Cost,  
 $= 3 \times 45 + 4 \times 30 + 1 \times 25 + 2 \times 80 + 4 \times 45 + 1 \times 75$   
 $= 135 + 120 + 25 + 160 + 180 + 75$   
 $= ₹695.$

### 3.5 MODI METHOD

#### Optimality Test

Once the initial basic feasible solution has been computed, the next step in the problem is to determine whether the solution obtained is optimum or not.

Optimality test can be conducted to any initial basic feasible solution of a TP provided such allocations has exactly  $m + n - 1$  non-negative allocations, where  $m$  is the number of origins and  $n$  is the number of destinations. Also, these allocations must be in independent positions.

To perform this optimality test, we shall discuss the Modified Distribution Method (MODI). The various steps involved in the MODI method for performing optimality test are as follows.

### MODI Method

**Step 1:** Find the initial basic feasible solution of a TP by using any one of the three methods.

**Step 2:** Find out a set of numbers  $u_i$  and  $v_j$  for each row and column satisfying  $u_i + v_j = C_{ij}$  for each occupied cell. To start with, we assign a number '0' to any row or column having maximum number of allocations. If this maximum number of allocations is more than one, choose any one arbitrarily.

**Step 3:** For each empty (unoccupied) cell, we find the sum  $u_i$  and  $v_j$  written in the bottom left corner of that cell.

**Step 4:** Find out for each empty cell the net evaluation value  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  and which is written at the bottom right corner of that cell. This step gives the optimality conclusion,

- (i) If all  $\Delta_{ij} > 0$  (i.e., all the net evaluation value) the solution is optimum and a unique solution exists.
- (ii) If  $\Delta_{ij} \geq 0$  then the solution is optimum, but an alternate solution exists.
- (iii) If at least one  $\Delta_{ij} < 0$ , the solution is not optimum. In this case, we go to the next step, to improve the total transportation cost.

**Step 5:** Select the empty cell having the most negative value of  $\Delta_{ij}$ . From this cell we draw a closed path by drawing horizontal and vertical lines with the corner cells occupied. Assign sign '+' and '-' alternately and find the minimum allocation from the cell having negative sign. This allocation should be added to the allocation having positive sign and subtracted from the allocation having negative sign.

**Step 6:** The previous step yields a better solution by making one (or more) occupied cell as empty and one empty cell as occupied. For this new set of basic feasible allocations, repeat from the Step (2) till an optimum basic feasible solution is obtained.

**Example 3.9:** Solve the following transportation problem.

		Destination				
		P	Q	R	S	Supply
Source	A	21	16	25	13	11
	B	17	18	14	23	13
	C	32	17	18	41	19
	Demand	6	10	12	15	43

Origin\Dest	P	Q	R	S	Supply	$P_I$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$	$P_{VI}$
A	21	16	25	13	11	3	-	-	-	-	-
B	17	18	14	23	13	3	4	4	4	-	-
C	32	17	18	41	19						
Demand	6	10	12	15	43	1	1	1	1	1	17
$P_I$	4	1	4	10↑							
$P_{II}$	15	1	4	18↑							
$P_{III}$	15↑	1	4	-							
$P_{IV}$	-	1	4	-							
$P_V$	-	17	18↑	-							
$P_{VI}$	-	17↑	-	-							

**Solution:** We first find the initial basic feasible solution by using VAM. Since  $\sum a_i = \sum b_j$ , the given TP is a balanced one. Therefore, there exists a feasible solution.

### NOTES

#### Check Your Progress

- What is a transportation problem?
- What must one know to achieve the objectives of a transportation problem?
- How are transportation problems presented?
- Give the mathematical presentation of a transportation problem.
- Define feasible, basic non-degenerate solutions of a transportation problem.

NOTES

Finally, we have the initial basic feasible solution as given in the following table.

		Destination			
		P	Q	R	S
Source	A	21	16	23	13
	B	17	18	14	23
	C	32	17	18	41

From this table, we see that the number of non-negative independent allocations is  $6 = m + n - 1 = 3 + 4 - 1$ .

Hence, the solution is non-degenerate basic feasible.

∴ The Initial Transportation Cost

$$= 11 \times 13 + 3 \times 14 + 4 \times 23 + 6 \times 17 + 17 \times 10 + 18 \times 9 = ₹711.$$

**To Find the Optimal Solution:** We apply the MODI method in order to determine the optimum solution. We determine a set of numbers  $u_i$  and  $v_j$  for each row and column, with  $u_i + v_j = C_{ij}$  for each occupied cell. To start with we give  $u_2 = 0$  as the second row has the maximum number of allocation.

Now, we find the sum  $u_i$  and  $v_j$  for each empty cell and enter at the bottom left corner of that cell.

$$C_{21} = u_2 + v_1 = 17 = 0 + v_1 = 17 \Rightarrow v_1 = 17$$

$$C_{23} = u_2 + v_3 = 14 = 0 + v_3 = 14 \Rightarrow v_3 = 14$$

$$C_{24} = u_2 + v_4 = 23 = 0 + v_4 = 23 \Rightarrow v_4 = 23$$

$$C_{14} = u_1 + v_4 = 13 = u_1 + 23 = 13 \Rightarrow u_1 = 10$$

$$C_{33} = u_3 + v_3 = 18 = u_3 + 14 = 18 \Rightarrow u_3 = 4$$

Next, we find the net evaluations  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each unoccupied cell and enter at the bottom right corner of that cell.

Initial Table

		$C_j$	4	3	0	0	-M	0	
$C_i$	$X_{ij}$	$X_{ij}$	$w_j$	$w_j$	$s_j$	$s_j$	$A_i$	$s_i$	Min $X_{ij}/w_j$
0	$S_1$	1	1	1	1	0	0	0	1
-M	$A_1$	1	0	1	0	-1	1	0	1
+0	$S_2$	1	-1	0	0	0	0	1	1/2
	$Z_1$	-M	0	-M	0	M	-M	0	
	$Z_1 - C_1$		-4	-M-3	0	M	0	0	
				↑					Min $\frac{X_{ij}}{w_j}$
+0	$S_1$	1/2	3/2	0	1	0	0	-1/2	1/2
-M	$A_1$	1/2	1/2	0	0	-1	1	-1/2	1
3	$w_2$	1/2	-1/2	1	0	0	0	1/2	-
	$Z_1$	$-\frac{1}{2}M - 3/2$	$\frac{1}{2}M - 3/2$	3	0	M	-M	$\frac{1}{2}M + 3/2$	
	$Z_1 - C_1$		$-\frac{M}{2} + \frac{3}{2}$	0	0	M	0	$\frac{M+3}{2}$	

Since all  $\Delta_{ij} > 0$ , the solution is optimal and unique. The optimum solution is given by,

$$X_{14} = 11; X_{21} = 6; X_{23} = 3; X_{24} = 4; X_{32} = 10; X_{33} = 9$$

The Minimum Transportation Cost,

$$= 11 \times 13 + 17 \times 6 + 3 \times 14 + 4 \times 23 + 10 \times 17 + 9 \times 18 = ₹711.$$

**Example 3.10:** Solve the following transportation problem starting with the initial solution obtained by VAM.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	2	2	2	1	3
$O_2$	10	8	5	4	7
$O_3$	7	6	6	8	5
Demand	4	3	4	4	15

**Solution:** Since  $\sum a_i = \sum b_j$ , the problem is a balanced TP. Therefore, there exists a feasible solution.

	$D_1$	$D_2$	$D_3$	$D_4$	Supply	$P_I$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$	$P_{VI}$
$O_1$	2	2	2	1	3	1	-	-	-	-	-
$O_2$	10	8	5	4	7	1	1	3	-	-	-
$O_3$	7	6	6	8	5	0	0	0	0	0	6
Demand	4	3	4	4	15						
$P_I$	5↑	4	4	3							
$P_{II}$	3	2	1	4↑							
$P_{III}$	3	2	1	-							
$P_{IV}$	7↑	6	6	-							
$P_V$	-	6↑	6	-							
$P_{VI}$	-	-	6	-							

Finally, the initial basic feasible solution is given as follows:

	$D_1$	$D_2$	$D_3$	$D_4$	Supply
$O_1$	2	2	2	1	3
$O_2$	10	8	5	4	7
$O_3$	7	6	6	8	5
Demand	4	3	4	4	15

Since the number of occupied cells =  $6 = m + n - 1$  and are also independent, there exists a non-degenerate basic feasible solution.

The Initial Transportation Cost,

$$= 3 \times 2 + 3 \times 5 + 4 \times 4 + 7 \times 1 + 6 \times 3 + 6 \times 1 = ₹68$$

To find the optimal solution, applying the MODI method, we determine a set of numbers  $u_i$  and  $v_j$  for each row and column, such that  $u_i + v_j = C_{ij}$  for each occupied cell. Since the third row has maximum number of allocations we give number  $u_3 = 0$ . The remaining numbers can be obtained as follows.

$$C_{31} = u_3 + v_1 = 7 = 0 + v_1 = 7 \Rightarrow v_1 = 7$$

$$C_{32} = u_3 + v_2 = 6 = 0 + v_2 = 6 \Rightarrow v_2 = 6$$

$$C_{33} = u_3 + v_3 = 6 = 0 + v_3 = 6 \Rightarrow v_3 = 6$$

$$C_{23} = u_2 + v_3 = 5 = u_2 + 6 = 5 \Rightarrow u_2 = -1$$

$$C_{13} = u_1 + v_3 = 4 = -1 + v_3 = 4 \Rightarrow v_3 = 5$$

$$C_{11} = u_1 + v_1 = 2 = u_1 + 7 = 2 \Rightarrow u_1 = -5$$

We find the sum  $u_i$  and  $v_j$  for each empty cell and enter at the bottom left corner of the cell. Next, we find the net evaluation  $\Delta_{ij}$  given by  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each empty cell and enter at the bottom right corner of the cell.

Initial Table

	$D_1$	$D_2$	$D_3$	$D_4$	$u_i$
$O_1$	2	2	2	1	
$O_2$	10	8	5	4	$u_2 = -1$
$O_3$	7	6	6	8	$u_3 = 0$
	$v_1 = 7$	$v_2 = 6$	$v_3 = 6$	$v_4 = 5$	

Since all  $\Delta_{ij} > 0$  the solution is optimum and unique. The solution is given by,  $X_{11} = 3; X_{23} = 3; X_{24} = 4; X_{31} = 1; X_{32} = 3; X_{33} = 1$

The Total Transportation Cost,

$$= 2 \times 3 + 3 \times 5 + 4 \times 4 + 7 \times 1 + 6 \times 3 + 6 \times 1 = ₹68.$$

### Degeneracy in Transportation Problem

In a TP, if the number of non-negative independent allocations is less than  $m + n - 1$ , where  $m$  is the number of origins (rows) and  $n$  is the number of destinations (columns) there exists a degeneracy. This may occur either at the initial stage or at the subsequent iteration.

To resolve this degeneracy, we adopt the following steps.

**Step 1:** Among the empty cell, we choose an empty cell having the least cost which is of an independent position. If this cell is more than one, choose any one arbitrarily.

**Step 2:** To the cell as chosen in step (1) we allocate a small positive quantity  $\epsilon > 0$ .

The cells containing ' $\epsilon$ ' are treated like other occupied cells and degeneracy is removed by adding one (more) accordingly. For this modified solution, we adopt the steps involved in MODI method till an optimum solution is obtained.

**Example 3.11:** Solve the transportation problem for minimization.

Sources	Destinations				Capacity
	1	2	3		
1	2	2	3	10	
2	4	1	2	15	
3	1	3	1	40	
Demand	20	15	30	65	

**Solution:** Since  $\sum a_i = \sum b_j$ , the problem is a balanced TP. Hence, there exists a feasible solution. We find the initial solution by North West Corner rule as given here.

	1	2	3	Capacity
1	2	2	3	10
2	4	1	2	15
3	1	3	1	40
Demand	20	15	30	

Since, the number of occupied cells =  $5 = m + n - 1$ , and all the allocations are independent, we get an initial basic feasible solution.

The Initial Transportation Cost,

$$= 10 \times 2 + 4 \times 10 + 5 \times 1 + 10 \times 3 + 1 \times 30 = 20 + 40 + 5 + 30 + 30 = ₹125.$$

**To Find the Optimal Solution (MODI Method):** We use the previous table to apply the MODI method. We find out a set of numbers  $u_i$  and  $v_j$  for which  $u_i + v_j = C_{ij}$  only for occupied cell. To start with, as the maximum number of allocations is 2 in more than one row and column, we choose arbitrarily column 1, and assign a number 0 to this column, i.e.,  $v_1 = 0$ . The remaining numbers can be obtained as follows.

$$C_{11} = u_1 + v_1 = 2 = u_1 + 0 = 2 \Rightarrow u_1 = 2$$

$$C_{21} = u_2 + v_1 = 4 \Rightarrow u_2 = 4 - 0 = 4$$

$$C_{22} = u_2 + v_2 = 1 \Rightarrow v_2 = 1 - u_2 = 1 - 4 = -3$$

$$C_{32} = u_3 + v_2 = 3 = u_3 = 3 - v_2 = 3 - (-3) = 6$$

$$C_{33} = u_3 + v_3 = 1 = v_3 = 1 - u_3 = 1 - 6 = -5$$

Initial Table

	1	2	3	$u_i$
1	2	2	3	$2 = u_1$
2	4	1	2	$4 = u_2$
3	6	3	1	$6 = u_3$
$v_j$	$v_1 = 0$	$v_2 = -3$	$v_3 = -5$	

We find the sum of  $u_i$  and  $v_j$  for each empty cell and write at the bottom left corner of that cell. Find the net evaluations  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each empty cell and enter at the bottom right corner of the cell.

The solution is not optimum as the cell (3, 1) has a negative  $\Delta_{ij}$  value. We improve the allocation by making this cell namely (3, 1) as an allocated cell. We draw a closed path from this cell and assign sign '+' and '-' alternately. From the cell having '-' sign we find the minimum allocation given by  $\text{Min}(10, 10) = 10$ . Hence, we get two occupied cells.

(2, 1) (3, 2) becomes empty and the cell (3, 1) is occupied and resulting in a degenerate solution. (Degeneracy in subsequent iteration).

Number of allocated cells =  $4 < m + n - 1 = 5$ .

**NOTES**

We get degeneracy. To resolve we add the empty cell (1, 2) and allocate  $\epsilon > 0$ . This cell namely (1, 2) is added as it satisfies the two steps for resolving the degeneracy. We assign a number 0 to the first row, namely  $u_1 = 0$  we get the remaining numbers as follows.

$$C_{11} = u_1 + v_1 = 2 \Rightarrow v_1 = 2 - u_1 = 2 - 0 = 2$$

$$C_{12} = u_1 + v_2 = 2 \Rightarrow v_2 = 2 - u_1 = 2 - 0 = 2$$

$$C_{31} = u_3 + v_1 = 1 \Rightarrow u_3 = 1 - v_1 = 1 - 2 = -1$$

$$C_{33} = u_3 + v_3 = 1 \Rightarrow v_3 = 1 - u_3 = 1 - (-1) = 2$$

$$C_{22} = u_2 + v_2 = 1 \Rightarrow u_2 = 1 - v_2 = 1 - 2 = -1$$

Next, we find the sum of  $u_i$  and  $v_j$  for the empty cell and enter at the bottom left corner of that cell and also the net evaluation  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each empty cell and enter at the bottom right corner of the cell.

Iteration Table

	1	2	3	$u_i$
1	⑩	⊕	2	0
2	1	⑤	1	-1
3	⑩	1	③①	-1
$v_j$	2	2	2	

The modified solution is given in the following table. This solution is also optimal and unique as it satisfies the optimality condition that all  $\Delta_{ij} > 0$ .

	1	2	3	Capacity
1	⑩	⊕	3	10
2		⑤	2	15
3	10	1	③①	40
Demand	②①	15	③①	65

$$X_{11} = 10; X_{22} = 15; X_{33} = 30; X_{12} = \epsilon; X_{31} = 10$$

$$\text{Total Cost} = 10 \times 2 + 2 \times \epsilon + 15 \times 1 + 10 \times 1 + 30 \times 1$$

$$= 75 + 2\epsilon = ₹75.$$

Example 3.12: Solve the following transportation problem whose cost matrix is given below.

		Destinations				
		A	B	C	D	Capacity
Origin	1	1	5	3	3	34
	2	3	3	1	2	15
	3	0	2	2	3	12
	4	2	7	2	4	19
Demand		21	25	17	17	80

Solution: Since  $\sum a_i = \sum b_j$ , the problem is a balanced transportation problem. Hence, there exists a feasible solution. We find the initial solution by North West Corner Rule.

		Destinations				
		A	B	C	D	Capacity
Origin	1	②①	⑬	3	3	34
	2	3	⑬	③	2	15
	3	0	2	⑫	3	12
	4	2	7	⑫	④	19
Demand		21	25	17	17	80

We get the total number of allocated cells =  $7 = 4 + 4 - 1$ . As all the allocations are independent, the solution is a non-degenerate solution.

Total Transportation Cost,

$$= 1 \times 21 + 5 \times 13 + 3 \times 12 + 3 \times 1 + 12 \times 2 + 2 \times 2 + 17 \times 4$$

$$= ₹221.$$

To Find the Optimal Solution (MODI Method): We determine a set of numbers  $u_i$  and  $v_j$  for each row and each column with  $u_i + v_j = C_{ij}$  for each occupied cell. To start with we give 0 to the third column as it has the maximum number of allocation.

Initial Table

	A	B	C	D	$u_i$
1	②①	⑬	3	3	3 = $u_1$
2	-1	⑫	+	③	1 = $u_2$
3	0	4	-	⑫	2 = $u_3$
4	0	4	3	⑫	2 = $u_4$
$v_j$	-2 = $v_1$	2 = $v_2$	0 = $v_3$	2 = $v_4$	

$$C_{23} = u_2 + v_3 = 1 \Rightarrow u_2 = 1 - 0 = 1$$

$$C_{33} = u_3 + v_3 = 2 \Rightarrow u_3 = 2 - 0 = 2$$

$$C_{43} = u_4 + v_3 = 2 \Rightarrow u_4 = 2 - 0 = 2$$

$$C_{34} = u_4 + v_4 = 4 \Rightarrow v_4 = 4 - 2 = 2$$

$$C_{22} = u_2 + v_2 = 3 \Rightarrow v_2 = 3 - u_2 = (3 - 1) = 2$$

$$C_{12} = u_1 + v_2 = 5 \Rightarrow u_1 = 5 - v_2 = (5 - 2) = 3$$

$$C_{11} = u_1 + v_1 = 1 \Rightarrow v_1 = 1 - u_1 = 1 - 3 = -2$$

We find the sum  $u_i$  and  $v_j$  for each empty cell and enter at the bottom left corner of that cell. We find the net evaluation  $\Delta_{ij} = C_{ij} - (u_i + v_j)$  for each empty cell and enter at the bottom right corner of that cell. The solution is not optimum as some of  $\Delta_{ij} < 0$ . We choose the most negative  $\Delta_{ij}$ , i.e., -2. There is a tie between the cell (1, 4) and (3, 2) but we choose the cell (3, 2) as it has the least cost. From this cell, we draw a closed path and assign '+' and '-' signs alternately and find the minimum allocation from the cell having '-' sign.

Thus, we get,  $\text{Min}(12, 12) = 12$ . Hence, one empty cell (3, 2) becomes occupied and two occupied cells (2, 2) (3, 3) becomes empty resulting in degeneracy (Degeneracy in subsequent iteration). By adding and subtracting this minimum allocation we get the modified allocation as given in the following table. For this modified allocations, we repeat the steps in the MODI method.

First Iteration Table

	A	B	C	D	
1	(21)	1	(13)	5	3
2	3	3	(15)	1	2
3	0	(12)	2	2	3
4	2	7	(2)	2	(17)

The number of allocations =  $6 < m + n - 1 = 7$ . We add the cell (3, 3) as it is the least cost empty cell which is of independent position. Give a small quantity  $\epsilon > 0$ . This removes degeneracy.

The modified allocation is given in the following table.

	A	B	C	D	$u_i$
1	(21)	5	3	3	5
2	3	3	1	2	1
3	0	(12)	(15)	3	2
4	2	7	(2)	4	2
$v_j$	-2	0	0	2	

The solution is not optimum. The next negative value of  $\Delta_{ij} = -4$  (the cell (1, 4)).

The minimum allocation is  $\text{Min}(13, \epsilon, 17) = \epsilon$ . Proceeding in the same manner we have the second iteration table given as follows.

Second Iteration Table

	A	B	C	D	$u_i$
1	(21)	(13- $\epsilon$ )	3	( $\epsilon$ )	0
2	3	3	1	2	0
3	0	(12+ $\epsilon$ )	(15)	3	-3
4	2	7	(2)	4	1
$v_j$	2	0	1	3	

As the solution is not optimum, we improve the solution by using the steps involved in the MODI method. The most negative value of  $\Delta_{ij} = -2$ . Minimum allocation is  $\text{Min}(13 - \epsilon, 15, 17 - \epsilon) = 13 - \epsilon$ .

Third Iteration Table

	A	B	C	D	
1	(21)	5	3	3	0
2	3	3	1	(13)	0
3	0	(12+ $\epsilon$ )	(2+ $\epsilon$ )	3	-1
4	2	7	(2)	4	1
$v_j$	2	0	1	3	

Improve the solution by adding and subtracting the new allocation given by  $\text{Min}(2 + \epsilon, 4) = (2 + \epsilon)$ .

Fourth Iteration Table

	A	B	C	D	$u_i$
1	(21)	5	3	3	3
2	3	3	1	(13)	2
3	0	(13- $\epsilon$ )	0	(2+ $\epsilon$ )	2
4	2	7	(2)	4	1
$v_j$	-1	0	-1	1	

NOTES

Since all  $\Delta_{ij} \geq 0$  the solution is optimum (alternate solution exists). The solution is given by,

$$X_{11} = 21; X_{14} = 13; X_{22} = 13 - \epsilon = 13; X_{24} = 2 + \epsilon = 2$$

$$X_{32} = 12 + \epsilon = 12; X_{43} = 17 + \epsilon = 17; X_{44} = 2 - \epsilon = 2$$

Total Transportation Cost,

$$= 21 \times 1 + 3 \times 13 + 3(13 - \epsilon) + 2(2 + \epsilon) + 2(12 + \epsilon) + 2(17 + \epsilon) + 4(2 - \epsilon) = 169 - \epsilon = ₹169.$$

**Example 3.13:** A company has three plants A, B and C; 3 warehouses X, Y, Z. The number of units available at the plants is 60, 70, 80 and the demand at X, Y, Z are 50, 80, 80 respectively. The unit cost of the transportation is given in the table.

	X	Y	Z
A	8	7	3
B	3	8	9
C	11	3	5

Find the allocation so that the total transportation cost is minimum.

Solution:

Plants	Warehouses			Capacity
	X	Y	Z	
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
Demand	50	80	80	210

Since  $\sum a_i = \sum b_j = 210$ , the problem is a balanced one. Hence, there exists a feasible solution. Let us find the initial solution by least cost method.

	X	Y	Z	Supply
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
Demand	50	80	80	210

Here, the least cost cell is not unique, i.e., the cells (2, 1) (1, 3) and (3, 2) have the least value 3. So, choose the cell arbitrarily. Let us choose the cell (2, 1) and allocate with magnitude  $\text{Min}(70, 50) = 50$ . This exhausts the first column. Now, delete this column. The reduced transportation table is given by,

	Y	Z	Supply
A	7	3	60
B	8	9	20
C	3	5	80
Demand	80	80	

Continuing in this manner, we finally arrive at the initial solution which is shown in the following table.

	X	Y	Z	Capacity
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
Demand	50	80	80	

The number of allocated cells is  $4 < m + n - 1 = 5$ , resulting in degeneracy (Initial stage). To remove this degeneracy we add an empty cell (3, 3) whose cost is minimum and is of independent position. Allocate to this cell a small quantity  $\epsilon > 0$ . Hence, we have the initial solution given in the following table:

	X	Y	Z	Capacity
A	8	7	3	60
B	3	8	9	70
C	11	3	5	80
Demand	50	80	80	

The total number of occupied cells =  $5 = m + n - 1 = 5$ . These are also of independent position. This solution is non-degenerate.

The solution is given by,

$$X_{13} = 60; X_{21} = 50; X_{23} = 20; X_{32} = 80; X_{33} = \epsilon$$

The Total Transportation Cost,

$$= 3 \times 60 + 3 \times 50 + 9 \times 20 + 3 \times 80 + 5 \times \epsilon = ₹750 + 5\epsilon = ₹750.$$

**To Find the Optimal Solution:** We apply the steps involved in the MODI method to the previous table. We find a set of numbers  $u_i$  and  $v_j$  for which  $u_i + v_j = C_{ij}$  is satisfied for each of the occupied cell. To start with we assign a number 0 to the third column (i.e.,  $v_3 = 0$ ) as it has the maximum number of allocations. The remaining numbers are obtained as follows.

	X	Y	Z	$u_i$
A	8	7	3	
B	3	8	9	
C	11	3	5	
$v_j$	-6	-2	0	

Since, all  $\Delta_{ij} > 0$  we have obtained an optimum solution.

The solution is given by,

$$X_{12} = 60; X_{21} = 50; X_{23} = 20; X_{32} = 80; X_{33} = \epsilon$$

$$\begin{aligned} \text{Total Transportation Cost} &= 3 \times 60 + 3 \times 50 + 9 \times 20 + 3 \times 80 + 5 \times \epsilon \\ &= ₹750 + 5 \epsilon = ₹750. \end{aligned}$$

### Unbalanced Transportation

The given TP is said to be unbalanced if  $\sum a_i \neq \sum b_j$ , i.e., if the total supply is not equal to the total demand.

There are two possible cases.

$$\text{Case (i): } \sum_{i=1}^m a_i < \sum_{j=1}^n b_j$$

If the total supply is less than the total demand, a dummy source (row) is included in the cost matrix with zero cost; the excess demand is entered as a rim requirement for this dummy source (origin). Hence, the unbalanced transportation problem can be converted into a balanced TP.

$$\text{Case (ii): } \sum_{i=1}^m a_i > \sum_{j=1}^n b_j$$

Therefore, the total supply is greater than the total demand. In this case, the unbalanced TP can be converted into a balanced TP by adding a dummy destination (column) with zero cost. The excess supply is entered as a rim requirement for the dummy destination.

**Example 3.14:** Solve the transportation problem when the unit transportation costs, demands and supplies are as given below:

Origins	Destination				Supply
	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	
O <sub>1</sub>	6	1	9	3	70
O <sub>2</sub>	11	5	2	8	55
O <sub>3</sub>	10	12	4	7	70
Demand	85	35	50	45	

**Solution:** Since the total demand  $\sum b_j = 215$  is greater than the total supply  $\sum a_i = 195$ , the problem is an unbalanced TP.

We convert this into a balanced TP by introducing a dummy origin O<sub>4</sub> with cost zero and giving supply equal to  $215 - 195 = 20$  units. Hence, we have the converted problem as follows:

Origins	Destination				Supply
	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	
O <sub>1</sub>	6	1	9	3	70
O <sub>2</sub>	11	5	2	8	55
O <sub>3</sub>	10	12	4	7	70
O <sub>4</sub>	0	0	0	0	20
Demand	85	35	50	45	215

As this problem is balanced, there exists a feasible solution to this problem. Using VAM we get the initial solution.

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply	P <sub>I</sub>	P <sub>II</sub>	P <sub>III</sub>	P <sub>IV</sub>	P <sub>V</sub>	P <sub>VI</sub>	P <sub>VII</sub>
O <sub>1</sub>	6 (65)	1 (5)	9	3	70	2	2	2	-	-	-	-
O <sub>2</sub>	11	5 (30)	2 (25)	8	55	3	3	3	3	6	-	-
O <sub>3</sub>	10	12	4 (25)	7 (45)	70	3	3	3	3	3	3	4
O <sub>4</sub>	0 (20)	0	0	0	20	0	-	-	-	-	-	-
Demand	85	35	50	45								
P <sub>I</sub>	6↑	1	2	3								
P <sub>II</sub>	4↑	4	2	3								
P <sub>III</sub>	-	4↑	2	4								
P <sub>IV</sub>	-	7↑	2	1								
P <sub>V</sub>	-	-	2	1								
P <sub>VI</sub>	-	-	4	7↑								
P <sub>VII</sub>	-	-	4	-								

The initial solution to the problem is given by,

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>
O <sub>1</sub>	65	5		
O <sub>2</sub>		30	25	
O <sub>3</sub>			25	45
O <sub>4</sub>	20			

There are 7 independent non-negative allocations equal to  $m + n - 1$ . Hence, the solution is a non-degenerate one.

The Total Transportation Cost,

$$\begin{aligned} &= 6 \times 65 + 5 \times 1 + 5 \times 30 + 2 \times 25 + 4 \times 25 + 7 \times 45 + 20 \times 0 \\ &= ₹1010. \end{aligned}$$

**To Find the Optimal Solution:** We apply the steps in the MODI method to the previous table.

	D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	u <sub>i</sub>
O <sub>1</sub>	65	5	0	3	0
O <sub>2</sub>	10	11	30	5	3
O <sub>3</sub>	12	10	12	4	4
O <sub>4</sub>	20	0	0	0	0
v <sub>j</sub>	6	1	-4	-5	5

NOTES

Since all  $\Delta_{ij} \geq 0$  the solution is not optimum and to solve we introduce the cell (3,1) as this cell has the most negative value of  $\Delta_{ij}$ . We modify the solution by adding and subtracting the minimum allocation given by  $\text{Min}(65, 30, 25)$ . While doing this, the occupied cell (3, 3) becomes empty.

First iteration table

	$D_1$	$D_2$	$D_3$	$D_4$	$u_i$
$O_1$	40	30	-2	3	6
$O_2$	10	5	50	7	10
$O_3$	25	5	7	45	10
$O_4$	20	-5	-8	-3	0
$v_j$	0	-5	-8	-3	

As the number of independent allocations are equal to  $m+n-1$ , we check the optimality.

Since, all  $\Delta_{ij} \geq 0$ , the solution is optimal and an alternate solution exists as  $\Delta_{14} = 0$ . Therefore, the optimum allocation is given by,

$$X_{11} = 40; X_{12} = 30; X_{22} = 5; X_{23} = 50; X_{31} = 25; X_{34} = 45; X_{41} = 20$$

The Optimum Transportation Cost,

$$= 6 \times 40 + 1 \times 30 + 5 \times 5 + 2 \times 50 + 10 \times 25 + 7 \times 45 + 0 \times 20 = ₹960.$$

**Example 3.15:** A product is produced by 4 factories  $F_1, F_2, F_3$  and  $F_4$ . Their unit production cost are ₹2, 3, 1 and 5 respectively. Production capacity of the factories are 50, 70, 40 and 50 units respectively. The product is supplied to 4 stores  $S_1, S_2, S_3$  and  $S_4$ , the requirements of which are 25, 35, 105 and 20 respectively.

Find the transportation plan such that the total production and transportation cost is minimum. Unit costs of transportation are given as follow:

Factory	Stores			
	$S_1$	$S_2$	$S_3$	$S_4$
$F_1$	2	4	6	11
$F_2$	10	8	7	5
$F_3$	13	3	9	12
$F_4$	4	6	8	3

**Solution:** We form the transportation table which consists of both production and transportation costs.

	$S_1$	$S_2$	$S_3$	$S_4$	Capacity
$F_1$	4	6	8	13	50
$F_2$	13	11	10	8	70
$F_3$	14	4	10	13	30
$F_4$	9	11	13	8	50
Demand	25	35	105	20	

Total capacity = 200 units

Total demand = 185 units

Therefore,  $\sum a_i > \sum b_j$ . Hence, the problem is unbalanced. We convert it into a balanced one by adding a dummy store  $S_5$  with cost 0 and the excess supply is given as the rim requirement to this store, namely (200-185) units.

	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$	Supply
$F_1$	4	6	8	13	0	50
$F_2$	13	11	10	8	0	70
$F_3$	14	4	10	13	0	30
$F_4$	9	11	13	8	0	50
Demand	25	35	105	20	15	200

The initial basic feasible solution is obtained by least cost method. We get the solution containing 8 non-negative independent allocations equal to  $m+n-1$ . So the solution is a non-degenerate solution.

The Total Transportation Cost,

$$= 4 \times 25 + 6 \times 5 + 8 \times 20 + 10 \times 50 + 8 \times 20 + 4 \times 30 + 13 \times 35 + 0 \times 15 = ₹1525.$$

**To Find the Optimal Solution:** We apply MODI method to the previous table as it has  $m+n-1$  independent non-negative allocations.

Initial Table

	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$	$u_i$
$F_1$	4	6	8	13	0	0
$F_2$	13	11	10	8	0	0
$F_3$	14	4	10	13	0	0
$F_4$	9	11	13	8	0	0
$v_j$	4	6	8	6	-5	5

The solution is not optimum as the cell (4, 4) is having a negative net evaluation value, i.e.,  $\Delta_{4,4} = -3$ . We draw a closed path from this cell and have a modified allocation by adding and subtracting the allocation  $\text{Min}(35, 20) = 20$ . This modified allocation is given in the table.

	$S_1$	$S_2$	$S_3$	$S_4$	$S_5$	$u_i$
$F_1$	4 (25)	6 (5)	8 (20)	3 0	0 5	0
$F_2$	13 7	11 8	10 (70)	8 (5)	0 -3	2
$F_3$	14 2	4 (50)	10	13 1	0 12	-2
$F_4$	9 0	11 0	13 (15)	8 20	0 (15)	5
$v$	4	6	8	3	-5	

Since all the values of  $\Delta_{ij} \geq 0$  the solution is optimum, but an alternate solution exists.

The optimum solution or the transportation plan is given by,

$$X_{11} = 25 \text{ units}; \quad X_{32} = 30 \text{ units}$$

$$X_{12} = 5 \text{ units}; \quad X_{43} = 15 \text{ units}$$

$$X_{13} = 20 \text{ units}; \quad X_{44} = 20 \text{ units}$$

$$X_{23} = 70 \text{ units}; \quad X_{45} = 15 \text{ units}$$

This is the surplus capacity that are not transported which are manufactured in factory  $F_4$ . The Optimum Production with Transportation Cost is as follows:

$$= 4 \times 25 + 6 \times 5 + 8 \times 20 + 10 \times 70 + 4 \times 30 + 8 \times 20 + 13 \times 15 + 0 \times 15 = ₹1465.$$

### 3.6 SPECIAL CASES

#### 3.6.1 Maximization Case in Transportation Problem

In this, the objective is to maximize the total profit for which the profit matrix is given. For this, first we have to convert the maximization problem into minimization by subtracting all the elements from the highest element in the given transportation table. This modified minimization problem can be solved in the usual manner.

Example 3.16: There are three factories  $A$ ,  $B$  and  $C$  which supply goods to four dealers  $D_1$ ,  $D_2$ ,  $D_3$  and  $D_4$ . The production capacities of these factories are 1000, 700 and 900 units per month respectively. The requirements from the dealers are 900, 800, 500 and 400 units per month respectively. The per unit return (excluding transportation cost) are ₹8, ₹7 and ₹9 at the three factories. The following table gives the unit transportation costs from the factories to the dealers.

	$D_1$	$D_2$	$D_3$	$D_4$
$A$	2	2	2	4
$B$	3	5	3	2
$C$	4	3	2	1

Determine the optimum solution to maximize the total returns.

Solution: Profit/Return-Transportation cost. With this we form a transportation table with profit.

	$D_1$	$D_2$	$D_3$	$D_4$
$A$	$8-2=6$	$8-2=6$	$8-2=6$	$8-4=4$
$B$	$7-3=4$	$7-5=2$	$7-3=4$	$7-2=5$
$C$	$9-4=5$	$9-3=6$	$9-2=7$	$9-1=8$

#### Profit Matrix

	$D_1$	$D_2$	$D_3$	$D_4$	Capacity
$A$	6	6	6	4	1000
$B$	4	2	4	5	700
$C$	5	6	7	8	900
Requirement	900	800	500	400	2600

This profit matrix is converted into its equivalent loss matrix by subtracting all the elements from the highest element, namely 8. Hence, we have the following loss matrix.

	$D_1$	$D_2$	$D_3$	$D_4$	Capacity
$A$	2	2	2	4	1000
$B$	4	6	4	3	700
$C$	3	2	1	0	900
Requirement	900	800	500	400	2600

We use VAM to get the initial basic feasible solution.

	$D_1$	$D_2$	$D_3$	$D_4$	Capacity	$P_I$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$
$A$	2 (200)	2 (800)	2	4	200 1000	0	0	0	2	2
$B$	4 (700)	6	4	3	0 700	1	0	2	4	-
$C$	3	2	1 (500)	0 (400)	900	1	1	-	-	-
Demand	900	800	500	400	2600					
$P_I$	1	0	1	3↑						
$P_{II}$	1	0	1	-						
$P_{III}$	2	4↑	-	-						
$P_{IV}$	2	-	-	-						
$P_V$	2↑	-	-	-						

NOTES

The initial solution is given in the following table:

	$D_1$	$D_2$	$D_3$	$D_4$	Capacity
A	2 (200)	2 (800)	2	4	1000
B	4 (700)	6	4	3	700
C	3	2	1 (500)	0 (400)	900
Demand	900	800	500	400	

Since the number of allocated cell =  $5 < m + n - 1 = 6$ , the solution is degenerate. To resolve this degeneracy we add an empty cell (1, 3) by allocating a non-negative quantity  $\epsilon$ .

This cell is the least cost cell and is of independent position. The initial basic feasible solution is given as follows:

Initial Table

	$D_1$	$D_2$	$D_3$	$D_4$
A	2 (200)	2 (800)	$\epsilon$	4
B	4 (700)	6	2	3
C	3	2	1 (500)	0 (400)

Number of allocations =  $6 = m + n - 1$  and the 6 allocations are in independent position. Hence, we can perform the optimality test using MODI method.

	$D_1$	$D_2$	$D_3$	$D_4$	$u_i$
A	(200) 2	(800) 2	$\epsilon$	4	0
B	(700) 4	6	2	3	2
C	3	2	(500) 1	(400) 0	-1
$u_i$	2	2	2	1	

Since all the net evaluation  $\Delta_{ij} > 0$  are non-negative the initial solution is optimum.

The optimum distribution is,

- A  $\rightarrow D_1 = 200$  units
- A  $\rightarrow D_2 = 800$  units
- A  $\rightarrow D_3 = \epsilon$  units
- B  $\rightarrow D_1 = 700$  units
- C  $\rightarrow D_3 = 500$  units
- A  $\rightarrow D_4 = 400$  units

Total profit or the Max. return,  
 $= 200 \times 6 + 6 \times 800 + 4 \times 700 + 7 \times 500 + 8 \times 400$   
 $= ₹15,500.$

Example 3.17: Solve the following transportation problem to maximize the profit.

Source	Destination				Supply
	A	B	C	D	
1	15	51	42	33	23
2	80	42	26	81	44
3	90	40	66	60	33
Demand	23	31	16	30	100

Solution: Since the given problem is to maximize the profit, we convert this into loss matrix and minimize it. For converting it into minimization type, we subtract all the elements from the highest element 90. Hence, we have the following loss matrix.

Source	Destination				Supply
	A	B	C	D	
1	75	39	48	57	23
2	10	48	64	9	44
3	0	50	24	30	33
Demand	23	31	16	30	100

Since  $\sum a_i = \sum b_j$ , there exists a feasible solution and is obtained by VAM.

	A	B	C	D	Supply	$P_I$	$P_{II}$	$P_{III}$	$P_{IV}$	$P_V$	$P_{VI}$
1	75	39 (23)	48	57	23	9	18	18	18	39	39
2	10 (6)	48 (8)	64	9 (30)	44	1	1	1	39	48	-
3	0 (17)	50	24 (16)	30	33	24	30	-	-	-	-
Demand	23	31	16	30							
$P_I$	10	9	24	21							
$P_{II}$	10	9	-	21							
$P_{III}$	65	9	-	48							
$P_{IV}$	-	9	-	48							
$P_V$	-	9	-	-							
$P_{VI}$	-	39	-	-							

The initial basic feasible solution is given in the following table.

	A	B	C	D	Capacity
1	2 (200)	2 (800)	2	4	1000
2	4 (700)	6	4	3	700
3	3	2	1 (500)	0 (400)	900
Demand	23	31	16	30	

As the number of independent allocated cells =  $6 = m + n - 1$  the solution is non-degenerate solution.

		A	B	C	D	$u_i$
1	19	75 50	39 23	48 23	57 18	9
2	6	10 5	48 14	64 50	9 30	0
3	17	0 55	50 12	24 16	30 -1	-10
$v_j$		10	48	14	9	

Since all the net evaluation  $\Delta_f > 0$  the solution is optimum and unique. The optimum solution is given by,

$$X_{12} = 23; X_{21} = 6; X_{24} = 30; X_{31} = 17; X_{33} = 16$$

$$\text{The Optimum Profit} = 23 \times 51 + 6 \times 80 + 42 \times 8 + 81 \times 30 + 90 \times 17 + 66 \times 16 = ₹7005.$$

### 3.6.2 Transshipment Problem

In a transportation problem, where shipments are allowed only between source-sink pairs, there is a possibility of existing points via which units of a goods/merchandise may be transhipped from a source to a sink. It is a strong assumption that shipments may be allowed between sources and between sinks, and also, inter-linking source-sink. Transportation models which have these additional features are called transshipment problems. Often, we see a gradual shift towards conversion from a transshipment problem to a transportation problem. This conversion procedure is of great significance as it broadens the applicability of algorithm as a solution for transportation problems. This conversion procedure can be well defined with the following example:

#### Transshipment Problem-to-Transportation Problem

An organic food company manufactures cereals in two cities, Leeds and Kent. The daily cereal production capacity at Leeds and Kent are 160 and 200 packets, respectively. Cereals are shipped by air to consumers in London and New York. The consumers in each city require 140 packets of cereals per day. However, due to the deregulation of air fares, the organic food company believes that it may be cheaper to fly some variety of cereals to Leeds or Dallas, and then do the final packaging fly the packets of cereals to London and New York (final destinations). The table given below shows the cost of flying one packet of the cereal between these cities:

From →	Leeds	Kent	London	Dallas	New York
Leeds	£ 0	—	£ 9	£ 14	£ 29
Kent	—	£ 0	£ 16	£ 13	£ 26
London	—	—	£ 0	£ 7	£ 18
Dallas	—	—	£ 7	£ 0	£ 17
New York	—	—	—	—	£ 0

Now, to minimize the total incurred cost of daily shipments of the cereals to its consumers, we first need to understand terminologies, such as source and sink. Source is a city that can send products, however cannot receive any product from any other city. Whereas, sink is a city that can receive products but cannot send to any other city.

So, in this example we can say, that Leeds and Kent are source, and Leeds and Dallas are transshipment points, and finally, London and New York are sinks (each with a daily requirement of 140 packets of cereals).

So, we see a mismatch in demand and supply with total supply equals to 156 and the total demand equals to 122.

Now, to solve this imbalance we need to create a dummy sink, with a demand of 34. We would now have 2 sources, 2 sinks, and 2 transshipment points. As discussed before, transshipment points can act in dual roles, both as sources and sinks. As there are no transportation costs from a transshipment point to itself, the primary objective to reduce costs remain intact.

Therefore, we should perform a reformulation and use the transshipment points as an optimal solution for imbalanced demand-supply as well as reduce transportation problem (costs) to ensure maximization of profits.

### 3.6.3 Transshipment Model

In a transshipment model, the objects are supplied from various specific sources to various specific destinations. It is also economic if the shipment passes via the transient nodes which are in between the sources and the destinations. It is different from transportation problem where the shipments are directly sent from a specific source to a specific destination, whereas in the transshipment problem the main goal is to reduce the total cost of shipments. Hence, the shipment passes via one or more intermediary nodes before it reaches its desired specific destination. Basically, there are two methods of evaluating transshipment problems as discussed below.

The following is the schematic illustration of the sources and destinations acting as transient nodes of a simple transshipment problem.

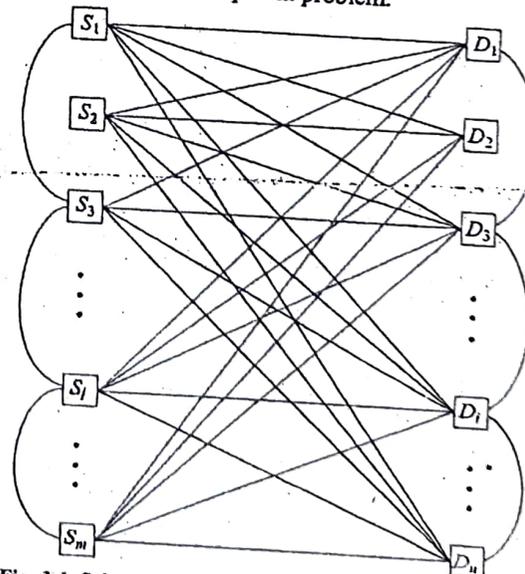


Fig. 3.1 Schematic Diagram of Simple Transshipment Model

NOTE

#### Check Your Progress

- Give reasons why the LPP solution technique is not used for solving transportation problem.
- List the approach used with transportation problems for determining the starting solution.
- Define the optimal solution to a transportation problem.
- State the necessary and sufficient conditions for the existence of a feasible solution to a transportation problem.
- What is the purpose of the MODI method?

NOTES

The figure shows the shipment of objects from source  $S_1$  to destination  $D_2$ . Shipment from source  $S_1$  can pass via  $S_2$  and  $D_1$  before it reaches the desired destination  $D_2$ . Because the shipment passes via the particular transient nodes, this arrangement is named as transshipment model. The goal of the transshipment problem is to discover the optimal shipping model so that the total transportation cost is reduced.

Figure 3.2 shows a different approach where the number of first starting nodes and also the number of last ending nodes is the sum of the total number of sources and destinations of the original problem. Let  $B$  be the buffer which should be maintained at every transient source and transient destination. Considering it as a balanced problem, buffer  $B$  at the least may be equal to the sum of total supplies or the sum of total demands. Therefore, a constant  $B$  is further added to all the starting nodes and the ending nodes as shown below:

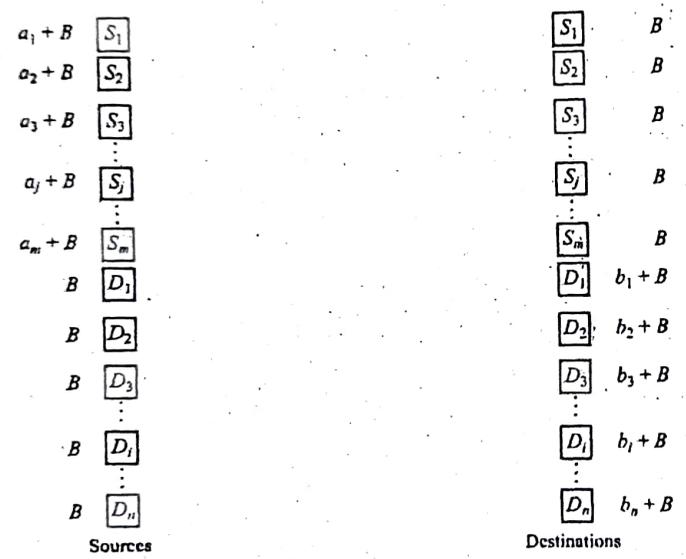


Fig. 3.2. Modified Version of Simple Transshipment Problem

Here in the modified version of simple transshipment model, the destinations  $D_1, D_2, D_3, \dots, D_{n-1}, \dots, D_n$  are incorporated as added starting nodes which basically acts as the transient nodes. Hence, these do not have the original supplies and at least the supply of every transient node must be equal to  $B$ . Therefore, every transient node is assigned  $B$  units of supply value. Also, the sources  $S_1, S_2, S_3, \dots, S_{m-1}, \dots, S_m$  are incorporated as added ending nodes which basically act as the transient nodes. These nodes too do not have the original demands but every transient node is assigned  $B$  units of demand value. To make it a balanced problem,  $B$  is further added to every starting node and to the ending nodes. Hence, the problem resembles a usual transportation problem and can be solved to obtain the optimum shipping plan.

**Example 3.18:** The following is the transshipment problem with 4 sources and 2 destinations. The supply values of the sources  $S_1, S_2, S_3$  and  $S_4$  are 100, 200, 150 and 350 units respectively. The demand values of destinations  $D_1$  and  $D_2$  are

350 and 450 units respectively. Transportation cost per unit between various defined sources and destinations are given in the following table. Solve the transshipment problem.

		Destination					
		$S_1$	$S_2$	$S_3$	$S_4$	$D_1$	$D_2$
Source	$S_1$	0	4	20	5	25	12
	$S_2$	10	0	6	10	5	20
	$S_3$	15	20	0	8	45	7
	$S_4$	20	25	10	0	30	6
	$D_1$	20	18	60	15	0	10
	$D_2$	10	25	30	23	4	0

**Solution:** In the above table the number of sources is 4 and the number of destinations is 2. Therefore, the total number of starting nodes and the ending nodes of the transshipment problem will be  $4 + 2 = 6$ . We also have,

$$B = \sum_{i=1}^n a_i = \sum_{j=1}^m b_j$$

The following is the detailed format of the transshipment problem after including transient nodes for the sources and destinations. Here, the value of  $B$  is added to all the rows and columns.

		Destination						Supply
		$S_1$	$S_2$	$S_3$	$S_4$	$D_1$	$D_2$	
Source	$S_1$	0	4	20	5	25	12	$100+800=900$
	$S_2$	10	0	6	10	5	20	$200+800=1000$
	$S_3$	15	20	0	8	45	7	$150+800=950$
	$S_4$	20	25	10	0	30	6	$350+800=1150$
	$D_1$	20	18	60	15	0	10	800
	$D_2$	10	25	30	23	4	0	800
		800	800	800	800	$350+800=1150$	$450+800=1250$	

The optimal solution and the corresponding total cost of transportation is ₹5,600. The allocations defined in the main diagonal cells are ignored. The diagrammatic representation of the optimal shipping pattern of the shipments related to the off-diagonal cells is shown in Figure 3.3:

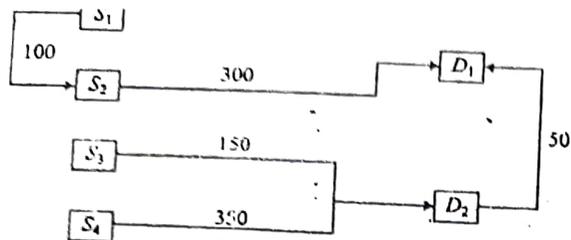


Fig. 3.3 Optimal Shipping Pattern

### 3.7 SUMMARY

- The transportation problem is one of the subclasses of LPP (Linear Programming Problem) in which the objective is to transport various quantities of a single homogeneous commodity that are initially stored at various origins to different destinations in such a way that the transportation cost is minimum.
- Any set of non-negative allocations ( $X_{ij} > 0$ ) which satisfies the row and column sum (rim requirement) is called a feasible solution.
- A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to  $m + n - 1$ , where  $m$  is the number of rows and  $n$  the number of columns in a transportation table.
- Any feasible solution to a transportation problem containing  $m$  origins and  $n$  destinations is said to be non-degenerate, if it contains  $m + n - 1$  occupied cells and each allocation is in independent positions.
- The allocations are said to be in independent positions if it is impossible to form a closed path. Closed path means by allowing horizontal and vertical lines and when all the corner cells are occupied.
- If a basic feasible solution contains less than  $m + n - 1$  non-negative allocations it is said to be degenerate.
- If demand is lesser than supply then dummy demand node is added to make it a balanced problem.
- Data is presented in tabular form. As a convention, origins are put on left side of the table with quantity to be supplied listed towards right side and demands are put on top with quantity of demand towards the bottom side.
- Unit cost of transportation is put at the top of every cell within a small box. Zero unit cost shows unshipped unit column in case supply is in excess of the demand. Similarly, a unit cost either penalty or zero shows shortage row in supplies that are lesser than demand.
- Basic cells are those that indicate positive values and non-basic cells have zero value for flow. According to transportation problem number of basic cells will be exactly  $m + n - 1$ .

- The dual values of every cell can be computed by setting the first to zero and appropriate order is used for dual equations for basic cells.
- In travelling salesman problem we cannot choose the element along the diagonal and this can be avoided by filling the diagonal with infinitely large elements. The travelling salesman problem is very similar to the assignment problem except that in the former case, there is an additional restriction that  $X_{ij}$  is so chosen that no city is visited twice before the tour of all the cities is completed.
- Optimal solution is a feasible solution (not necessarily basic) which minimizes the total cost.
- The solution of a Transportation Problem (TP) can be obtained in two stages, namely initial solution and optimum solution.
- Initial solution can be obtained by using any one of the three methods, viz.
  - (i) North West Corner Rule (NWCR)
  - (ii) Least Cost Method or Matrix Minima Method
  - (iii) Vogel's Approximation Method (VAM)
- VAM is preferred over the other two methods, since the initial basic feasible solution obtained by this method is either optimal or very close to the optimal solution.
- The cells in the transportation table can be classified as occupied cells and unoccupied cells. The allocated cells in the transportation table are called occupied cells and empty cells in a transportation table are called unoccupied cells.
- The improved solution of the initial basic feasible solution is called optimal solution which is the second stage of solution that can be obtained by MODI (Modified Distribution Method).
- Once the initial basic feasible solution has been computed, the next step in the problem is to determine whether the solution obtained is optimum or not.
- Optimality test can be conducted to any initial basic feasible solution of a TP provided such allocations has exactly  $m + n - 1$  non-negative allocations, where  $m$  is the number of origins and  $n$  is the number of destinations. Also, these allocations must be in independent positions.
- In a TP, if the number of non-negative independent allocations is less than  $m + n - 1$ , where  $m$  is the number of origins (rows) and  $n$  is the number of destinations (columns) there exist degeneracy. This may occur either at the initial stage or at the subsequent iteration.
- The TP is said to be unbalanced if  $\sum a_i \neq \sum b_j$ , i.e., if the total supply is not equal to the total demand.
- If the total supply is less than the total demand, a dummy source (row) is included in the cost matrix with zero cost; the excess demand is entered as a rim requirement for this dummy source (origin). Hence, the unbalanced transportation problem can be converted into a balanced TP.

## NOTES

- In a transportation problem, where shipments are allowed only between source-sink pairs, there is a possibility of existing points via which units of a goods/merchandise may be transhipped from a source to a sink. It is a strong assumption that shipments may be allowed between sources and between sinks, and also, inter-linking source-sink.
- In a transshipment model, the objects are supplied from various specific sources to various specific destinations. It is also economic if the shipment passes via the transient nodes which are in between the sources and the destinations.
- Transshipment model is different from transportation problem where the shipments are directly sent from a specific source to a specific destination, whereas in the transshipment problem the main goal is to reduce the total cost of shipments.

## 3.8 KEY TERMS

- **Transportation problem:** A problem for transportation of various quantities of a single homogeneous commodity, initially stored at various origins to different destinations at the minimum cost
- **Feasible solution:** A set of non-negative allocations where some quantity is transferred from an origin  $i$  to a destination  $j$ , ( $X_{ij} > 0$ ) and satisfies the row and column sum is a feasible solution
- **Basic feasible solution:** A feasible solution where number of non-negative allocations is equal to  $m + n - 1$ , where  $m$  is the number of rows and  $n$  is the number of columns in a transportation table
- **Non-degenerate basic feasible solution:** Any feasible solution to a transportation problem containing  $m$  origins and  $n$  destinations is known as non-degenerate, if it contains  $m + n - 1$  occupied cells and each allocation is in independent positions
- **Degenerate basic feasible solution:** A basic feasible solution that contains less than  $m + n - 1$  non-negative allocations

## 3.9 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Transportation problem deals with transportation of various quantities of a single homogeneous commodity initially stored at various origins to different destinations at the minimum cost.
2. One must know: (i) Amount and location of available supplies, (ii) Quantity demanded at destination and (iii) Costs of transporting one unit of commodity from various origins to various destinations.
3. Transportation problems are presented in tabular form with  $m$  rows (as origins) and  $n$  columns (as destinations). Every cell of this table has cost as  $C_{ij}$  of transporting one unit of product from  $i$ th origin to  $j$ th destination and  $X_{ij}$  which is the quantity transported from  $i$ th origin to  $j$ th destination.
4. If there are  $m$  origins and  $n$  destinations and  $C_{ij}$  be the cost of transporting one unit of product from  $i$ th origin to  $j$ th destination and  $X_{ij}$  be the quantity transported

from  $i$ th origin to  $j$ th destination, and  $Z$  is the cost given as:

$$Z = \sum_{i=1}^m \sum_{j=1}^n C_{ij} X_{ij} \text{ where } 1 \leq i \leq m \text{ and } 1 \leq j \leq n.$$

5. **Feasible solution:** Any set of non-negative allocations ( $X_{ij} \geq 0$ ) which satisfies the row and column sum (rim requirement) is called a feasible solution.  
**Basic feasible solution:** A feasible solution is called a basic feasible solution if the number of non-negative allocations is equal to  $m + n - 1$ , where  $m$  is the number of rows  $n$  the number of columns in a transportation table.  
**Non-degenerate basic feasible solution:** Any feasible solution to a transportation problem containing  $m$  origins and  $n$  destinations is said to be non-degenerate, if it contains  $m + n - 1$  occupied cells and each allocation is in independent positions.
6. As there are  $m + n - 1$  equations in a transportation problem with  $m$  origins and  $n$  destinations, by adding an artificial variable to each equation, a large number of variables are involved.
  - (i) If the problem has  $m$  sources and  $n$  destinations and  $m + n - 1$  equations can be formed. Hence, computation may exceed the capacity of the computer. So LPP solution is not made use for solving a TP.
  - (ii) The coefficient  $X_{ij}$  in the constraints are all in unity. For such a technique, transportation technique is easier than simplex method.
  - (iii) TP is minimization of objective function, whereas, simplex method is suitable for maximization problem.
7. The following approaches are used:
  - (i) North West Corner Rule
  - (ii) Least Cost Method (Matrix Minima)
  - (iii) Vogel's Approximation Method
8. The basic feasible solution to a transportation problem is said to be optimal if it minimizes the total transportation cost.
9. The necessary and sufficient condition for the existence of a feasible solution is a solution that satisfies all conditions of supply and demand.
10. The purpose of MODI method is to get the optimal solution of a transportation problem.

## 3.10 QUESTIONS AND EXERCISES

## Short-Answer Questions

1. What is an unbalanced transportation problem?
2. How will you convert an unbalanced transportation problem into a balanced one?
3. List the merits and limitations of the North West Corner rule.
4. Vogel's Approximation Method results in the most economical initial basic feasible solution. How?
5. How will you identify if a transportation problem has an alternate optimal solution or not?

- What are the numbers of non-basic variables in a balanced transportation problem?
- While dealing with North West Corner rule, when does one move to the next cell in next column?
- What is the coefficient of  $X_{ij}$  of constraints in a transportation problem?
- What kind of solution would you get when net change in value of all unoccupied cells is non-negative?
- What is an initial basic feasible solution?
- When does degeneracy occur in an  $m \times n$  transportation problem?

**Long-Answer Questions**

- What do you understand by transportation model?
- Define feasible solution, basic solution, non-degenerate solution, optimal solution in a transportation problem.
- Explain the following briefly with examples:
  - North West Corner Rule
  - Least Cost Method
  - Vogel's Approximation Method
- Explain degeneracy in a transportation problem. Describe a method to resolve it.
- What do you mean by an unbalanced transportation problem? Explain the process of converting an unbalanced transportation problem into a balanced one.
- Give the mathematical formulation of a transportation problem.
- Write an algorithm to solve a transportation problem.
- Obtain the initial solution for the following transportation problem using
  - North West Corner Rule
  - Least Cost Method
  - VAM

		Destination			
		A	B	C	Supply
Source	1	2	7	4	5
	2	3	3	1	8
	3	5	4	7	7
	4	1	6	2	14
Demand		7	9	18	34

[Ans.

- $X_{11} = 5, X_{21} = 2, X_{22} = 6, X_{32} = 3, X_{33} = 4, X_{43} = 14$  and the transportation cost ₹. 102.
- $X_{12} = 2, X_{13} = 3, X_{22} = 8, X_{23} = 7, X_{41} = 7, X_{43} = 7$  and transportation cost ₹. 83.
- $X_{11} = 5, X_{22} = 8, X_{32} = 7, X_{41} = 2, X_{42} = 2, X_{43} = 10$  and transportation cost ₹. 80.]

- Solve the following transportation problem where the cell entries denote the unit transportation costs.

		Destination				
		A	B	C	D	Supply
Origin	P	5	4	2	6	20
	Q	8	3	5	7	30
	R	5	9	4	6	50
Demand		10	40	20	30	100

[Ans.  $X_{12} = 10, X_{13} = 10, X_{22} = 30, X_{31} = 10, X_{33} = 10, X_{34} = 30$ . The optimum transportation cost is ₹. 420.]

- Solve the following transportation problem.

		Destination			
		1	2	3	Capacity
Source	1	2	2	3	10
	2	4	1	2	15
	3	1	3	1	40
Demand		20	15	30	

[Ans.  $X_{12} = 10, X_{22} = 15, X_{31} = 20, X_{33} = 15, X_{34} = 5$ . The transportation cost is ₹. 100.]

- Find the minimum transportation cost.

		Warehouse				
		D <sub>1</sub>	D <sub>2</sub>	D <sub>3</sub>	D <sub>4</sub>	Supply
Factory	F <sub>1</sub>	19	30	50	10	7
	F <sub>2</sub>	70	30	40	60	9
	F <sub>3</sub>	40	8	70	30	18
Demand		5	8	7	14	

[Ans.  $X_{11} = 5, X_{14} = 2, X_{22} = 2, X_{23} = 7, X_{32} = 6, X_{34} = 12$  and the minimum transportation cost = ₹. 743.]

- Solve the following transportation problem using Vogel's Approximation Method.

		Warehouse						
		A	B	C	D	E	F	Available
Factory	1	9	12	9	6	9	10	5
	2	7	3	7	7	5	5	6
	3	6	5	9	11	3	11	2
	4	6	8	11	2	2	10	9
Requirement		4	4	6	2	4	2	

[Ans.  $X_{13} = 5, X_{22} = 4, X_{34} = 2, X_{31} = 1, X_{33} = 6, X_{43} = 1, X_{41} = 3, X_{44} = 2, X_{45} = 4$  and the min. transportation cost is ₹. 112.]

NOTES

13. Solve the following transportation problem.

		Destination				
		A	B	C	D	Supply
Source	1	1	2	3	4	6
	2	4	3	2	0	8
	3	0	2	2	1	10
Demand		4	6	8	6	

[Ans.  $X_{12} = 6, X_{23} = 2, X_{34} = 6, X_{31} = 4, X_{32} = 0, X_{33} = 6$   
The min. transportation cost is ₹. 28.]

14. Solve the following transportation problem.

		Destination				
		A	B	C	D	Supply
Source	1	11	20	7	8	50
	2	21	16	20	12	40
	3	8	12	8	9	70
Demand		30	25	35	40	

[Ans.  $X_{13} = 35, X_{14} = 15, X_{24} = 10, X_{23} = 30, X_{31} = 30, X_{32} = 25, X_{34} = 15$   
min. transportation cost = ₹. 1,160.]

15. Solve the following transportation problem to maximize the profit.

		Destination				
		A	B	C	D	Supply
Source	1	40	25	22	33	100
	2	44	35	30	30	30
	3	38	38	28	30	70
Demand		40	20	60	30	

[Ans.  $X_{11} = 20, X_{14} = 30, X_{15} = 50, X_{21} = 20, X_{23} = 10, X_{32} = 20, X_{33} = 50$   
and the optimum profit is ₹. 5,130]

MODULE IV

3.11 FURTHER READING

Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.

Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.

Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.

## UNIT 4 ASSIGNMENT PROBLEM

### Structure

- 4.0 Introduction
- 4.1 Unit Objectives
- 4.2 Basics of Assignment Problem
  - 4.2.1 Mathematical Formulation of the Assignment Problem
- 4.3 Applications and Solutions by Hungarian Method
  - 4.3.1 Hungarian Method Procedure
  - 4.3.2 Maximization in Assignment Problem
- 4.4 Job Sequence Problem
  - 4.4.1 Unbalanced Assignment Problem
- 4.5 Variations of the Assignment Problem
- 4.6 Summary
- 4.7 Key Terms
- 4.8 Answers to 'Check Your Progress'
- 4.9 Questions and Exercises
- 4.10 Further Reading

NOTE:

### 4.0 INTRODUCTION

This unit will cover the basic concepts of an assignment problem, which can be presented in the form of  $n \times n$  cost matrix of real numbers. For solving an assignment problem, the Hungarian method is used. You will also learn about maximization in assignment problems.

You will also learn about the job sequence problem. To solve a job sequence problem, the assignment problem has to be balanced. An assignment problem is balanced if the cost matrix is a square matrix, otherwise it is termed as unbalanced. To make an unbalanced assignment problem balanced, dummy rows or dummy columns are added with all entries as zeroes.

### 4.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Define assignment problem
- Solve an assignment problem using the Hungarian method
- Understand how to maximize an assignment problem
- Solve a job sequence problem

### 4.2 BASICS OF ASSIGNMENT PROBLEM

The assignment problem is one of the fundamental combinatorial optimization problems. It helps to find a maximum weight identical in nature in a weighted bipartite graph. The assignment problem is also termed as a special case of transportation problem.

NOTES

Suppose there are  $n$  jobs to be performed and  $n$  persons are available for doing these jobs. Assume that each person can do each job at a time, though with varying degrees of efficiency. Let  $C_{ij}$  be the cost if the  $i$ th person is assigned to the  $j$ th job. The solution to the problem is to find an assignment (which job should be assigned to which person on one-one basis) so that the total cost of performing all jobs is minimum. Problems of this kind are known as assignment problems.

The assignment problem can be stated in the form of  $n \times n$  cost matrix  $[C_{ij}]$  of real numbers as given in the following table:

		Jobs				
		1	2	3...	$j...$	$n$
Persons	1	$C_{11}$	$C_{12}$	$C_{13}...$	$C_{1j}...$	$C_{1n}$
	2	$C_{21}$	$C_{22}$	$C_{23}...$	$C_{2j}...$	$C_{2n}$
	3	$C_{31}$	$C_{32}$	$C_{33}...$	$C_{3j}...$	$C_{3n}$
	$i$	$C_{i1}$	$C_{i2}$	$C_{i3}...$	$C_{ij}...$	$C_{in}$
	$n$	$C_{n1}$	$C_{n2}$	$C_{n3}...$	$C_{nj}...$	$C_{nn}$

### 4.2.1 Mathematical Formulation of the Assignment Problem

Mathematically, the assignment problem can be stated as follows:

$$\text{Minimize } Z = \sum_{i=1}^n \sum_{j=1}^n C_{ij} x_{ij} \quad i=1, 2, \dots, n$$

$$j=1, 2, \dots, n$$

Subject to the restrictions:

$$x_{ij} = \begin{cases} 1 & \text{if the } i\text{th person is assigned } j\text{th job} \\ 0 & \text{if not} \end{cases}$$

$$\sum_{j=1}^n x_{ij} = 1 \quad (\text{one job is done by the } i\text{th person})$$

$$\text{and } \sum_{i=1}^n x_{ij} = 1 \quad (\text{only one person should be assigned the } j\text{th job})$$

Where  $x_{ij}$  denotes that the  $j$ th job is to be assigned to the  $i$ th person.

## 4.3 APPLICATIONS AND SOLUTIONS BY HUNGARIAN METHOD

### 4.3.1 Hungarian Method Procedure

The solution of an assignment problem can be arrived at by using the Hungarian method. The steps involved in this method are as follows:

**Step 1:** Prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row (column) with zero cost element.

**Step 2:** Subtract the minimum element in each row from all the elements of the respective rows.

**Step 3:** Further modify the resulting matrix by subtracting the minimum element of each column from all the elements of the respective columns. Thus, obtain the modified matrix.

**Step 4:** Then, draw minimum number of horizontal and vertical lines to cover all zeroes in the resulting matrix. Let the minimum number of lines be  $N$ . Now, there are two possible cases.

**Case (i)** If  $N = n$ , where  $n$  is the order of matrix, then an optimal assignment can be made. So make the assignment to get the required solution.

**Case (ii)** If  $N < n$ , then proceed to step 4.

**Step 5:** Determine the smallest uncovered element in the matrix (element not covered by  $N$  lines). Subtract this minimum element from all uncovered elements and add the same element at the intersection of horizontal and vertical lines. Thus, the second modified matrix is obtained.

**Step 6:** Repeat Steps (3) and (4) until we get the Case (i) of Step 4.

**Step 7:** (To make zero assignment) Examine the rows successively until a row with exactly single zero is found. Circle (O) this zero to make the assignment. Then mark a cross (x) over all zeroes if lying in the column of the circled zero showing that they can not be considered for future assignment. Continue in this manner until all the zeroes have been examined. Repeat the same procedure for the columns also.

**Step 8:** Repeat the step 6 successively until one of the following situation arises:

**Case (i)** If no unmarked zero is left, then the process ends.

**Case (ii)** If there lies more than one of the unmarked zero in any column or row, then circle one of the unmarked zeroes arbitrarily and mark a cross in the cells of remaining zeroes in its row or column. Repeat the process until no unmarked zero is left in the matrix.

**Step 9:** Thus, exactly one marked circled zero in each row and each column of the matrix is obtained. The assignment corresponding to these marked circled zeroes will give the optimal assignment.

**Example 4.1:** Using the following cost matrix and determine (i) Optimal job assignment, and (ii) The cost of assignments.

		Jobs				
		1	2	3	4	5
Mechanics	A	10	3	3	2	8
	B	9	7	8	2	7
	C	7	5	6	2	4
	D	3	5	8	2	4
	E	9	10	9	6	10

Step 1: Select the smallest element in each row and subtract this smallest element from all the other elements in its row.

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ A & 8 & 1 & 1 & 0 & 6 \\ B & 7 & 5 & 6 & 0 & 5 \\ C & 5 & 3 & 4 & 0 & 2 \\ D & 1 & 3 & 6 & 0 & 2 \\ E & 3 & 4 & 3 & 0 & 4 \end{matrix}$$

Step 2: Subtract the minimum element from each column and subtract this element from all the elements in its column. With this we get the first modified matrix.

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ A & 7 & 0 & 0 & 0 & 4 \\ B & 6 & 4 & 5 & 0 & 3 \\ C & 4 & 2 & 3 & 0 & 0 \\ D & 0 & 2 & 5 & 0 & 0 \\ E & 2 & 3 & 2 & 0 & 2 \end{matrix}$$

In this modified matrix, we draw the minimum number of lines to cover all zeroes (horizontal and vertical).

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ A & 7 & 0 & 0 & 0 & 4 \\ B & 6 & 4 & 5 & 0 & 3 \\ C & 4 & 2 & 3 & 0 & 0 \\ D & 0 & 2 & 5 & 0 & 0 \\ E & 2 & 3 & 2 & 0 & 2 \end{matrix}$$

Number of lines drawn to cover all zeroes is  $N = 4$ .

The order of matrix is  $n = 5$ .

Hence,  $N < n$ , less than order of matrix.

Now, we get the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding it to the element at the point of intersection of lines.

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ A & 9 & \textcircled{1} & \times & 2 & 6 \\ B & 6 & 2 & 3 & \textcircled{0} & 3 \\ C & 4 & \times & 1 & \times & \textcircled{0} \\ D & \textcircled{0} & \times & 3 & \times & \times \\ E & 2 & 1 & \textcircled{0} & \times & 2 \end{matrix}$$

Number of lines drawn to cover all zeroes is  $N = 5$

The order of matrix is  $n = 5$

Hence,  $N = n$ , i.e.; order of matrix. Now, we determine the optimum assignment.

Assignment

$$\begin{matrix} & 1 & 2 & 3 & 4 & 5 \\ A & 9 & \textcircled{1} & \times & 2 & 6 \\ B & 6 & 2 & 3 & \textcircled{0} & 3 \\ C & 4 & \times & 1 & \times & \textcircled{0} \\ D & \textcircled{0} & \times & 3 & \times & \times \\ E & 2 & 1 & \textcircled{0} & \times & 2 \end{matrix}$$

The first row contains more than one zero. So, proceed to the second row. It has exactly one zero. The corresponding cell is  $(B, 4)$ . Circle this zero thus, making an assignment. Mark  $(\times)$  for all other zeroes in its column to show that they cannot be used for making other assignments. Now, row 5 has a single zero in the cell  $(E, 3)$ . Make an assignment in this cell and cross the 2nd zero in the 3rd column.

Now, row 1 has a single zero in the column 2, i.e., in the cell  $(A, 2)$ .

Make an assignment in this cell and cross the other zeroes in the 2nd column. This leads to a single zero in the column 1 of the cell  $(D, 1)$ . Make an assignment in this cell and cross the other zeroes in the 4th row. Finally, we have a single zero left in the 3rd row making an assignment in the cell  $(C, 5)$ . Thus, we have the following assignment.

Table: Optimal Assignment and Optimum Cost of Assignment

Job	Mechanic	Cost
1	D	3
2	A	3
3	E	9
4	B	2
5	C	4

Rs 21

Therefore,  $1 \rightarrow D, 2 \rightarrow A, 3 \rightarrow E, 4 \rightarrow B, 5 \rightarrow C$  with minimum cost Rs 21.

Example 4.2: A company has 5 jobs to be done on five machines. Any job can be done on any machine. The cost of doing the jobs in different machines are as follows. Assign the jobs for different machines so as to minimize the total cost.

Jobs	Machines				
	A	B	C	D	E
1	13	8	16	18	19
2	9	15	24	9	12
3	12	9	4	4	4
4	6	12	10	8	13
5	15	17	18	12	20

NOTES

**Solution:** We form the first modified matrix by subtracting the minimum element from all the elements in the respective rows and same with respective columns.

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 \begin{array}{l}
 1 \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \end{bmatrix} \\
 2 \begin{bmatrix} 0 & 6 & 15 & 0 & 3 \end{bmatrix} \\
 3 \begin{bmatrix} 8 & 5 & 0 & 0 & 0 \end{bmatrix} \\
 4 \begin{bmatrix} 0 & 6 & 4 & 2 & 7 \end{bmatrix} \\
 5 \begin{bmatrix} 3 & 5 & 6 & 0 & 8 \end{bmatrix}
 \end{array}
 \end{array}$$

Since each column has the minimum element 0. We have the first modified matrix. Now we draw the minimum number of lines to cover all zeroes.

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 \begin{array}{l}
 1 \begin{bmatrix} 5 & 0 & 8 & 10 & 11 \end{bmatrix} \\
 2 \begin{bmatrix} 0 & 6 & 15 & 0 & 3 \end{bmatrix} \\
 3 \begin{bmatrix} 8 & 5 & 0 & 0 & 0 \end{bmatrix} \\
 4 \begin{bmatrix} 0 & 6 & 4 & 2 & 7 \end{bmatrix} \\
 5 \begin{bmatrix} 3 & 5 & 6 & 0 & 8 \end{bmatrix}
 \end{array}
 \end{array}$$

The number of lines drawn to cover zeroes is  $N=4$ , which is less than the order of matrix which is  $n=5$ .

We find the second modified matrix by subtracting the smallest uncovered element from all the uncovered elements and add to the element which is at the point of intersection of lines.

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 \begin{array}{l}
 1 \begin{bmatrix} 5 & 0 & 5 & 10 & 8 \end{bmatrix} \\
 2 \begin{bmatrix} 0 & 6 & 12 & 0 & 0 \end{bmatrix} \\
 3 \begin{bmatrix} 1 & 8 & 0 & 0 & 0 \end{bmatrix} \\
 4 \begin{bmatrix} 0 & 6 & 1 & 2 & 7 \end{bmatrix} \\
 5 \begin{bmatrix} 3 & 5 & 3 & 0 & 5 \end{bmatrix}
 \end{array}
 \end{array}$$

The number of lines drawn to cover all zeroes is  $N=5=n=5$ , which is the order of matrix. Hence, we can form an assignment.

**Assignment**

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 \begin{array}{l}
 1 \begin{bmatrix} 5 & \textcircled{0} & 5 & 10 & 8 \end{bmatrix} \\
 2 \begin{bmatrix} \text{X} & 6 & 12 & \text{X} & \textcircled{0} \end{bmatrix} \\
 3 \begin{bmatrix} 1 & 8 & \textcircled{0} & 3 & \text{X} \end{bmatrix} \\
 4 \begin{bmatrix} \textcircled{0} & 6 & 1 & 2 & 4 \end{bmatrix} \\
 5 \begin{bmatrix} 3 & 5 & 3 & \textcircled{0} & 5 \end{bmatrix}
 \end{array}
 \end{array}$$

All the five jobs have been assigned to 5 different machines.

Here, the optimal assignment is as follows:

Job	Machine
1	B
2	E
3	C
4	A
5	D

Minimum (Total cost) =  $8+12+4+6+12 = \text{Rs } 42$ .

**Example 4.3:** Four different jobs can be done on four different machines and down time costs are prohibitively high for change overs. The following matrix gives the cost in rupees of producing job  $i$  on machine  $j$ :

Jobs	Machine			
	$M_1$	$M_2$	$M_3$	$M_4$
$J_1$	5	7	11	6
$J_2$	8	5	9	6
$J_3$	4	7	10	7
$J_4$	10	4	8	3

How should the jobs be assigned to the various machines so that the total cost is minimized?

**Solution:** First we form a modified matrix by subtracting the least element in the respective row and respective column.

$$\begin{array}{c}
 M_1 \quad M_2 \quad M_3 \quad M_4 \\
 \begin{array}{l}
 J_1 \begin{bmatrix} 0 & 2 & 6 & 1 \end{bmatrix} \\
 J_2 \begin{bmatrix} 3 & 0 & 4 & 1 \end{bmatrix} \\
 J_3 \begin{bmatrix} 0 & 3 & 6 & 3 \end{bmatrix} \\
 J_4 \begin{bmatrix} 7 & 1 & 5 & 0 \end{bmatrix}
 \end{array}
 \end{array}$$

Since the third column has no zero element, we subtract the smallest element 4 from all the elements.

$$\begin{array}{c}
 M_1 \quad M_2 \quad M_3 \quad M_4 \\
 \begin{array}{l}
 J_1 \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix} \\
 J_2 \begin{bmatrix} 3 & 0 & 0 & 1 \end{bmatrix} \\
 J_3 \begin{bmatrix} 0 & 3 & 2 & 3 \end{bmatrix} \\
 J_4 \begin{bmatrix} 7 & 1 & 1 & 0 \end{bmatrix}
 \end{array}
 \end{array}$$

$$\begin{array}{c}
 M_1 \quad M_2 \quad M_3 \quad M_4 \\
 \begin{array}{l}
 J_1 \begin{bmatrix} 0 & 2 & 2 & 1 \end{bmatrix} \\
 J_2 \begin{bmatrix} 3 & 0 & 0 & 1 \end{bmatrix} \\
 J_3 \begin{bmatrix} 0 & 3 & 2 & 3 \end{bmatrix} \\
 J_4 \begin{bmatrix} 7 & 1 & 1 & 0 \end{bmatrix}
 \end{array}
 \end{array}$$

NOTES

The number of lines drawn to cover all zeroes is  $N = 3$ , which is less than the order of matrix is  $n = 4$ .

Hence, we form the second modified matrix, by subtracting the smallest uncovered element from all the uncovered elements and adding to the element which is at the point of intersection of lines.

$$\begin{array}{c} M_1, M_2, M_3, M_4 \\ \begin{array}{c} J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \left[ \begin{array}{cccc} 0 & 1 & 1 & 1 \\ 5 & 0 & 0 & 2 \\ 0 & 2 & 1 & 3 \\ 8 & 0 & 0 & 0 \end{array} \right]
 \end{array}$$

$N = 3 < n = 4$

$$\begin{array}{c} M_1, M_2, M_3, M_4 \\ \begin{array}{c} J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \left[ \begin{array}{cccc} 0 & 0 & 0 & 0 \\ 5 & 0 & 0 & 2 \\ 0 & 1 & 0 & 2 \\ 8 & 0 & 0 & 0 \end{array} \right]
 \end{array}$$

$N = 4 = n = 4 = \text{Order of matrix.}$

$$\begin{array}{c} M_1, M_2, M_3, M_4 \\ \begin{array}{c} J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \left[ \begin{array}{cccc} \infty & \infty & \infty & 0 \\ 5 & 0 & \infty & 2 \\ 0 & 1 & \infty & 2 \\ 8 & \infty & 0 & \infty \end{array} \right]
 \end{array}$$

Hence, we can make an assignment.

Since no rows and no columns have single zero, we have different assignment (Multiple solution).

Optimal Assignment

Jobs	Machines
$J_1$	$M_4$
$J_2$	$M_2$
$J_3$	$M_1$
$J_4$	$M_3$

Minimum (Total cost)

$6 + 5 + 4 + 8 = ₹ 23.$

Alternate Solution

$$\begin{array}{c} M_1, M_2, M_3, M_4 \\ \begin{array}{c} J_1 \\ J_2 \\ J_3 \\ J_4 \end{array} \left[ \begin{array}{cccc} 0 & 8 & 8 & 8 \\ 5 & 0 & 8 & 2 \\ 8 & 1 & 0 & 2 \\ 8 & 8 & 8 & 0 \end{array} \right]
 \end{array}$$

$J_1 \rightarrow M_1; J_2 \rightarrow M_2; J_3 \rightarrow M_3; J_4 \rightarrow M_4$

Minimum (Total cost)  $5 + 5 + 10 + 3 = ₹ 23.$

Example 4.4: Solve the following assignment problem in order to minimize the total cost. The following cost matrix given below gives the assignment cost when different operators are assigned to various machines.

	Operators					
	I	II	III	IV	V	
Machines	A	30	25	33	35	36
	B	23	29	38	23	26
	C	30	27	22	22	22
	D	25	31	29	27	32
	E	27	29	30	24	32

Solution: We form the first modified matrix by subtracting the least element from all the elements in the respective rows and then in the respective columns.

$$\begin{array}{c} I, II, III, IV, V \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \left[ \begin{array}{ccccc} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 8 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{array} \right]
 \end{array}$$

Since each column has the minimum element 0, the first modified matrix is obtained. We draw the minimum number of lines to cover all zeroes.

$$\begin{array}{c} I, II, III, IV, V \\ \begin{array}{c} A \\ B \\ C \\ D \\ E \end{array} \left[ \begin{array}{ccccc} 5 & 0 & 8 & 10 & 11 \\ 0 & 6 & 15 & 0 & 3 \\ 0 & 5 & 0 & 0 & 0 \\ 0 & 6 & 4 & 2 & 7 \\ 3 & 5 & 6 & 0 & 8 \end{array} \right]
 \end{array}$$

The number of lines drawn to cover all zeroes is  $N = 4 < \text{The order of matrix } n = 4$ . Hence, we form the second modified matrix by subtracting the smallest

NOTES

uncovered element from the remaining uncovered elements and add to the element which is at the point of intersection of lines.

$$\begin{array}{c}
 I \quad II \quad III \quad IV \quad V \\
 \begin{array}{l}
 A \left[ \begin{array}{ccccc}
 5 & 0 & 5 & 10 & 8 \\
 0 & 6 & 12 & 0 & 0 \\
 11 & 8 & 0 & 3 & 0 \\
 0 & 6 & 1 & 2 & 4 \\
 3 & 5 & 3 & 0 & 5
 \end{array} \right]
 \end{array}$$

$N = 5$ , i.e, the number of lines drawn to cover all zeroes = The order of matrix. Hence, we can make the assignment.

Here  $N = 5 = n = 5$  = Order of matrix.

$$\begin{array}{c}
 I \quad II \quad III \quad IV \quad V \\
 \begin{array}{l}
 A \left[ \begin{array}{ccccc}
 5 & \textcircled{0} & 5 & 10 & 8 \\
 \cancel{0} & 6 & 12 & \cancel{0} & \textcircled{0} \\
 11 & 8 & \textcircled{0} & 3 & \cancel{0} \\
 \textcircled{0} & 6 & 1 & 2 & 4 \\
 3 & 5 & 3 & \textcircled{0} & 5
 \end{array} \right]
 \end{array}$$

The optimum assignment is,

Operators	Machines
I	D
II	A
III	C
IV	E
V	B

The optimum cost is given by,  
 $25 + 25 + 22 + 24 + 26 = \text{Rs } 122.$

4.3.2 Maximization in Assignment Problem

In this, the objective is to maximize the profit. To solve this, we first convert the given profit matrix into the loss matrix by subtracting all the elements from the highest element of the given profit matrix. For this converted loss matrix we apply the steps in Hungarian method to get the optimum assignment.

**Example 4.5:** The owner of a small machine shop has four mechanics available to assign jobs for the day. Five jobs are offered with expected profit for each mechanic on each job which are as follows:

Jobs

	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
Mechanics 3	87	64	111	71	81
4	48	64	87	77	80

Find using the assignment method the assignment of mechanics to the job that will result in a maximum profit. Which job should be declined ?

**Solution:** The given profit matrix is not a square matrix as the number of jobs is not equal to the number of mechanics. Hence, we introduce a dummy mechanic 5 with all the elements 0.

Jobs

	A	B	C	D	E
1	62	78	50	101	82
2	71	84	61	73	59
Mechanics 3	87	92	111	71	81
4	48	64	87	77	80
5	0	0	0	0	0

Now, we convert this profit matrix into loss matrix by subtracting all the elements from the highest element 111.

Loss Matrix

	A	B	C	D	E
1	49	33	61	10	29
2	40	27	50	38	52
3	24	19	0	40	30
4	63	47	24	34	31
5	111	111	111	111	111

We subtract the smallest element from all the elements in the respective row.

	A	B	C	D	E
1	39	23	51	0	19
2	13	0	23	11	25
3	24	19	0	40	30
4	39	23	0	10	7
5	0	0	0	0	0

Since each column has minimum element as zero, we draw minimum number of lines to cover all zeroes.

	A	B	C	D	E
1	39	30	58	0	19
2	13	0	23	11	25
3	24	19	0	30	30
4	32	23	0	10	7
5	0	7	0	0	0

Here, number of line drawn to cover all zeroes is  $N = 4 < \text{Order of matrix}$ .

From the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered element and add to the element which is at the point of intersection of lines.

	A	B	C	D	E
1	39	30	58	0	19
2	6	0	23	4	18
3	17	19	0	33	23
4	32	23	0	3	0
5	0	7	7	0	0

Here,  $N = 5 = n = 5 = \text{Order of matrix}$ .

We make the assignment.

	A	B	C	D	E
1	39	30	58	⊙	19
2	6	⊙	23	4	18
3	17	19	⊙	33	23
4	32	23	⊗	3	⊙
5	⊙	7	7	⊗	⊗

The optimum assignment is,

Jobs	Mechanics
A	5
B	2
C	3
D	1
E	4

Since the 5th mechanic is a dummy, job A is assigned to the 5th mechanic, this job is declined.

The maximum profit is given by  $84 + 111 + 101 + 80 = \text{Rs } 376$ .

**Example 4.6:** A marketing manager has 5 salesmen and there are 5 sales districts. Considering the capabilities of the salesmen and the nature of 12 districts, the estimates made by the marketing manager for the sales per month (in 1000 rupees) for each salesmen in each district would be as follows.

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesmen to the districts that will result in the maximum sales.

**Solution:** We are given the profit matrix. To maximize the profit, first we convert it into loss matrix which can be minimized. To convert it into loss matrix, we subtract all the elements from the highest element 41. Subtract the smallest element from all the elements in the respective rows and columns, to get the first modified matrix.

Loss Matrix

	A	B	C	D	E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

	A	B	C	D	E
1	8	2	0	12	0
2	0	16	12	19	4
3	0	14	8	11	4
4	19	3	0	5	5
5	11	7	0	5	1

	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	9	1	0	0	5
5	11	5	0	0	1

NOTES

We now draw minimum number of lines to cover all zeroes.

	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	9	1	0	0	5
5	11	5	0	0	1

$N = 4 < n = 5$ .

We subtract the smallest uncovered element from the remaining uncovered elements and add to the elements at the point of intersection of lines, to get the second modified matrix.

	A	B	C	D	E
1	9	0	1	8	0
2	0	13	12	14	3
3	0	11	8	6	3
4	9	0	0	0	4
5	11	4	0	0	0

Again,  $N = 4 < n = 5$ . Repeat the above step.

	A	B	C	D	E
1	12	0	1	8	0
2	0	10	9	11	0
3	0	8	5	3	0
4	12	0	0	0	4
5	14	4	0	0	0

$N = 5 = n = 5 =$  Order of matrix. Hence, we make the assignment.

Assignment

	A	B	C	D	E
1	12	⊙	1	8	⊗
2	⊙	10	9	11	⊗
3	⊗	8	5	3	⊙
4	12	⊗	⊙	⊗	4
5	14	4	⊗	⊙	⊗

Since no row or column has single zero, we get a multiple solution.

(i) The optimum assignment is:

$1 \rightarrow B, 2 \rightarrow A, 3 \rightarrow E, 4 \rightarrow C, 5 \rightarrow D$ .

With maximum profit  $(38 + 40 + 37 + 41 + 35) = \text{Rs } 191$ .

(ii) The optimum assignment is:

$1 \rightarrow B, 2 \rightarrow A, 3 \rightarrow E, 4 \rightarrow D, 5 \rightarrow C$ .

Maximum profit  $(38 + 40 + 37 + 36 + 40) = \text{Rs } 191$ .

### 4.4 JOB SEQUENCE PROBLEM

Job sequencing is basically the planning of the jobs in sequential manner and is an essential part of any work. Without proper planning and scheduling one can not achieve the desired output and profit. For sequencing a job, generally the two techniques are used termed as Priority Rules and Johnson's Rules. Priority rules give the guidelines for properly sequencing the job, where as Johnson's rule is used to minimize the completion time for a set of jobs to be done on two different machines. Using these rules one can assign jobs and maximize product and profit.

#### Basic Characteristics of Job Sequencing

1. Only one single job should be scheduled for a machine at a time.
2. Do not stop the process in between before completion.
3. New processing can be started after the completion of the previous processing.
4. Any job is scheduled for processing as per the order and due date requirements.
5. If the jobs are transferred from one machine to another due to some reason, then the time involved in transferring the jobs is considered negligible.

**Priority Rules:** These rules are used to get specific guidelines for job sequencing. The rules do not consider job setup cost and time while analysing processing times. In it job processing time and due dates are given importance because the due dates are fixed to give delivery in time to the customers. The rules are very useful for process-focussed amenities, for example health clinics, print shops and manufacturing industries. Hence, priority rules minimize the time for completing a job, sequences the jobs in the organization, checks if any job is late and maximizes resource utilization. The most popular priority rules are as follows:

- **First Come First Serve (FCFS):** The job to be processed first is the job that turned up first in the organization.
- **Earliest Due Date (EDD):** The job to be processed first is the job that has earliest due date.
- **Shortest Processing Time (SPT):** The job to be processed first and completed is the job that is shortest in nature; in other words the job can be processed in short time.
- **Longest Processing Time (LPT):** The job to be processed first is the job that is very important or of high priority though it can take longer processing time.

- **Critical Ratio (CR):** The job to be processed first is analysed on the basis of critical ratio, which is an index number calculated from time remaining until due date divided by the remaining work time.

**Johnson's Rule:** This rule is applied to minimize the completion time for a set of jobs that are to be processed on two different machines or at two consecutive work stations. The main objectives of the rules are,

- To minimize the processing time while sequencing a set of jobs on two different machines or work stations.
- To minimize the complete idle time on the processing machines.
- To minimize the flow time of the job, i.e., from the start of the first job until the completion of the last job.

**Necessary Conditions for Johnson's Rules:** The necessary conditions to efficiently complete the processing of the jobs are as follows:

- Knowledge about job time for each job at the specific work station.
- Job time must not depend on sequencing of jobs.
- All the jobs to follow the predefined work sequence.
- Avoid job priority.

**Four Steps Johnson's Rule:** The following are the important four steps in Johnson's rule:

**Step 1:** List all the jobs and the processing time of each machine to which these jobs are scheduled.

**Step 2:** Choose the job which has the shortest processing time. If the shortest time has been scheduled on the first machine or work station then the job is selected first for processing. In case the shortest time is scheduled on the second machine or work station then the job is processed at the end.

**Step 3:** After scheduling the job for processing go to Step 4.

**Step 4:** Repeat Step 2 again to schedule the processing of remaining jobs and fill the sequence columns towards the centre till all the jobs are scheduled.

The following example will help you to understand how the sequences are scheduled.

For example, there are five jobs to be done at a factory and each job must be processed through two work stations at two different machines, drill machine and lathe machine. Using Johnson's rule we can schedule the sequence of jobs.

The time (in hours) for processing each job is given in the following table:

Jobs	Work Station 1 (Drill)	Work Station 2 (Lathe)
A	5	2
B	3	6
C	9	4
D	12	8
E	8	14

Using the Steps of Johnson's Rule, the job processing sequences are scheduled as follows:

**Step 1:** In the given table, the job with the shortest processing time is job A, in work station 2 (with a time of 2 hours). Because it is at the second work station, schedule A last.

					A
--	--	--	--	--	---

**Step 2:** Next shortest time is of job B (with a time of 3 hours). Because it is at the first work station, schedule it at first priority and eliminate it from the list.

B					A
---	--	--	--	--	---

**Step 3:** The next shortest time is of job C (with a time of 4 hours), but it is at the second work station. Therefore, place it at last before A.

B			C		A
---	--	--	---	--	---

**Step 4:** There is a tie between job D (with a time of 8 hours at work station 2) and job E (with a time of 8 hours at work station 1) for the shortest remaining job. Because job E is at the first work station, so place it first after job B. Then place job D in the last sequencing position. You will get the job sequence schedule as follows:

B	E		D		C		A
---	---	--	---	--	---	--	---

The final sequential times at both the work stations will be:

Jobs	B	E	D	C	A
Work Station 1 (Drill)	3	8	12	9	5
Work Station 2 (Lathe)	6	14	8	4	2

#### 4.4.1 Unbalanced Assignment Problem

An assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and the number of columns are not equal. To make it balanced, we add a dummy row or a dummy column with all the entries as zeroes.

**Example 4.7:** There are four jobs to be assigned to the machines. Only one job could be assigned to one machine. The amount of time in hours required for the jobs in a machine is given in the following matrix.

Jobs	Machines				
	A	B	C	D	E
1	4	3	6	2	7
2	10	12	11	14	16
3	4	3	2	1	5
4	8	7	6	9	6

Find an optimum assignment of jobs to the machines to minimize the total processing time and also find for which machine no job is assigned. What is the total processing time to complete all the jobs?

Solution: Since the cost matrix is not a square matrix, the problem is unbalanced. We add a dummy job 5 with corresponding entries zeroes.

Modified Matrix

	A	B	C	D	E
1	4	3	6	2	7
2	10	12	11	14	16
3	4	3	2	1	5
4	8	7	6	9	6
5	0	0	0	0	0

We subtract the smallest element from all the elements in the respective row.

	A	B	C	D	E
1	2	1	4	0	5
2	0	2	1	4	6
3	3	2	1	0	4
4	2	1	0	3	0
5	0	0	0	0	0

Since each column has minimum element as zero, we draw minimum number of lines to cover all zeroes.

	A	B	C	D	E
1	2	1	4	0	5
2	0	2	1	4	6
3	3	2	1	0	4
4	2	1	0	3	0
5	0	0	0	0	0

The number of lines to cover all zeroes is  $N = 4 < \text{The order of matrix}$ . We form the second modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and add to the element at the point of intersection of lines.

	A	B	C	D	E
1	2	0	3	0	4
2	0	1	0	4	5
3	3	1	0	0	3
4	3	1	0	4	0
5	1	0	0	1	0

Here, the number of lines drawn to cover all zeroes is  $N = 5 = n = 5 = \text{Order of matrix}$ . Therefore, we can make the assignment,

	A	B	C	D	E
1	2	⓪	3	ⓧ	4
2	⓪	1	ⓧ	4	5
3	3	1	ⓧ	⓪	3
4	3	1	⓪	4	ⓧ
5	1	ⓧ	ⓧ	1	⓪

Optimum Assignment

Jobs	Machines
1	B
2	A
3	D
4	C

For machine E no job is assigned.

Optimum (minimum) cost =  $4 + 11 + 1 + 6 = 22$ .

Example 4.8: A company has 4 machines to do 3 jobs. Each job can be assigned to one and only one machine. The following is the cost of each job on each machine. Determine the job assignments which will minimize the total cost.

		Machines			
		W	X	Y	Z
Jobs	A	18	24	28	32
	B	8	13	17	18
	C	10	15	19	22

Solution: Since the cost matrix is not a square matrix we add a dummy row D with all the elements 0.

		W	X	Y	Z
A	18	24	28	32	
B	8	13	17	18	
C	10	15	19	22	
D	0	0	0	0	

Subtract the minimum element in each row from all the elements in its row.

		W	X	Y	Z
A	0	6	10	14	
B	0	5	9	10	
C	0	5	9	12	
D	0	0	0	0	

Since each column has minimum element we draw minimum number of lines to cover all zeroes.

$$\begin{array}{c}
 W \quad X \quad Y \quad Z \\
 A \begin{bmatrix} 0 & 6 & 10 & 14 \\ 0 & 5 & 9 & 10 \\ 0 & 5 & 9 & 12 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{array}$$

The number of lines drawn to cover all zeroes is  $N = 2 <$  The order of matrix. We form a second modified matrix.

$$\begin{array}{c}
 W \quad X \quad Y \quad Z \\
 A \begin{bmatrix} 0 & 1 & 5 & 9 \\ 0 & 0 & 4 & 5 \\ 0 & 0 & 4 & 7 \\ 0 & 0 & 0 & 0 \end{bmatrix}
 \end{array}$$

Here,  $N = 3 < n = 4$ .

Again, we subtract the smallest uncovered element from all the uncovered elements and add to the element at the point of intersection.

$$\begin{array}{c}
 W \quad X \quad Y \quad Z \\
 A \begin{bmatrix} 0 & 1 & 1 & 5 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 3 \\ 0 & 4 & 0 & 0 \end{bmatrix}
 \end{array}$$

Here,  $N = 4 = n = 4 =$  Order of matrix. Hence, we make an assignment.

**Assignment**

$$\begin{array}{c}
 W \quad X \quad Y \quad Z \\
 A \begin{bmatrix} \textcircled{0} & 1 & 1 & 5 \\ \textcircled{X} & \textcircled{0} & \textcircled{X} & 1 \\ \textcircled{H} & \textcircled{H} & \textcircled{0} & 3 \\ 9 & 4 & \textcircled{H} & \textcircled{0} \end{bmatrix} \begin{array}{l} A \rightarrow W \\ D \rightarrow Z \\ B \rightarrow X \\ C \rightarrow Y \end{array}
 \end{array}
 \quad \text{or} \quad
 \begin{array}{c}
 W \quad X \quad Y \quad Z \\
 A \begin{bmatrix} \textcircled{0} & 1 & 1 & 5 \\ \textcircled{H} & \textcircled{H} & \textcircled{0} & 1 \\ \textcircled{H} & \textcircled{0} & \textcircled{H} & 3 \\ 9 & 4 & \textcircled{H} & \textcircled{0} \end{bmatrix} \begin{array}{l} A \rightarrow W \\ D \rightarrow Z \\ B \rightarrow Y \\ C \rightarrow X \end{array}
 \end{array}$$

Since  $D$  is a dummy job, machine  $Z$  has assigned no job.

Therefore, optimum cost =  $18 + 13 + 19 =$  Rs 50

**4.5 VARIATIONS OF THE ASSIGNMENT PROBLEM**

**Unbalanced Assignment Problem**

Any assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and columns are not equal. To make it balanced, we add a dummy row or dummy column with all the entries as zero.

**Example 4.9:** There are four jobs to be assigned to five machines. Only one job can be assigned to one machine. The amount of time in hours required for the jobs per machine are given in the following matrix.

Jobs	Machines				
	A	B	C	D	E
1	4	3	6	2	7
2	10	12	11	14	16
3	4	3	2	1	5
4	8	7	6	9	6

Find an optimum assignment of jobs to the machines to minimize the total processing time and also find out for which machine no job is assigned. What is the total processing time to complete all the jobs?

**Solution:** Since the cost matrix is not a square matrix, the problem is unbalanced. We add a dummy job 5 with corresponding entries zero.

**Modified Matrix**

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 1 \begin{bmatrix} 4 & 3 & 6 & 2 & 7 \\ 10 & 12 & 11 & 14 & 16 \\ 4 & 3 & 2 & 1 & 5 \\ 8 & 7 & 6 & 9 & 6 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
 \end{array}$$

We subtract the smallest element from all the elements in the respective rows.

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 1 \begin{bmatrix} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 2 & 1 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
 \end{array}$$

Since each column has minimum element as zero, we draw minimum number of lines to cover all zeros.

$$\begin{array}{c}
 A \quad B \quad C \quad D \quad E \\
 1 \begin{bmatrix} 2 & 1 & 4 & 0 & 5 \\ 0 & 2 & 1 & 4 & 6 \\ 3 & 2 & 1 & 0 & 4 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}
 \end{array}$$

The number of lines to cover all zeros =  $4 <$  the order of matrix. We form the 2nd modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element at the point of intersection of lines.

Your Progress  
 balanced  
 assignment? How  
 it be balanced?  
 is the  
 ence of an  
 nate optional  
 tion  
 blished?  
 do you convert  
 maximization  
 ment  
 lem into a  
 nization one?  
 will you solve  
 assignment  
 tem if profit is  
 maximized?  
 f-Instructional  
 Material

NOTES

	A	B	C	D	E
1	2	0	3	0	4
2	0	1	0	4	5
3	3	1	0	0	3
4	3	1	0	4	0
5	1	0	0	1	0

Here the number of lines drawn to cover all zeros = 5 = Order of matrix. Therefore, we can make the assignment as follows:

	A	B	C	D	E
1	2	0	3	0	4
2	0	1	0	4	5
3	3	1	0	0	3
4	3	1	0	4	0
5	1	0	0	1	0

Optimum Assignment

1		B	3
2		A	10
3		D	1
4		C	6

For machine E, no job is assigned.

Optimum (minimum) cost = 3 + 10 + 1 + 6 = ₹20.

**Example 4.10:** A company has 4 machines to do 3 jobs. Each job can be assigned to only one machine. The cost of each job on each machine is given below. Determine the job assignments that will minimize the total cost.

	Machines			
	W	X	Y	Z
A	18	24	28	32
Jobs B	8	13	17	18
C	10	15	19	22

**Solution:** Since the cost matrix is not a square matrix, we add a dummy row D with all the elements 0.

	W	X	Y	Z
A	18	24	28	32
B	8	13	17	18
C	10	15	19	22
D	0	0	0	0

Subtract the minimum element in each row from all the elements in its row.

	W	X	Y	Z
A	0	6	10	14
B	0	5	9	10
C	0	5	9	12
D	0	0	0	0

Since each column has a minimum element 0, we draw minimum number of lines to cover all zeros.

	W	X	Y	Z
A	0	6	10	14
B	0	5	9	10
C	0	5	9	12
D	0	0	0	0

∴ The number of lines drawn to cover all zeros = 2 < Less than the order of matrix, hence we form a second modified matrix.

	W	X	Y	Z
A	0	1	5	9
B	0	0	4	5
C	0	0	4	7
D	0	0	0	0

Here, N = 3 < n = 4 = Less than the order of matrix.

Again we subtract the smallest uncovered element from all the uncovered elements and add to the element at the point of intersection

	W	X	Y	Z
A	0	1	1	5
B	0	0	0	1
C	0	0	0	3
D	0	0	0	0

Here, N = 4 = n = 4 = order of matrix. Hence, we make an assignment.

Assignment

A	0	1	1	5	A → W	A	X	1	1	5	A → W
B	X	0	X	1	B → X	B	X	X	X	1	B → Y
C	X	X	0	3	C → Y	C	X	X	X	3	C → X
D	9	4	X	0	D → Z	D	9	4	X	X	D → Z

Since D is a dummy job, machine Z is assigned no job.

Therefore, optimum cost = 18 + 13 + 19 = Rs 50.

Maximization In Assignment Problem

In this, the objective is to maximize the profit. To solve this, we first convert the given profit matrix into the loss matrix by subtracting all the elements from the highest

get the optimum assignment. We apply the steps in Hungarian method to

Example 4.11: The owner of a small machine shop has four mechanics available to assign jobs for the day. Five jobs are offered with expected profit for each mechanic on each jobs, which are as follows:

		Jobs				
		A	B	C	D	E
Mechanics	1	62	78	50	111	82
	2	71	84	61	73	59
	3	87	92	111	71	81
	4	48	64	87	77	80

By using the assignment method, find the assignment of mechanics to the job that will result in maximum profit. Which job should be declined?

Solution: The given profit matrix is not a square matrix as the number of jobs is not equal to the number of mechanics. Hence, we introduce a dummy mechanic 5 with all the elements 0.

		Jobs				
		A	B	C	D	E
Mechanics	1	62	78	50	111	82
	2	71	84	61	73	59
	3	87	92	111	71	81
	4	48	64	87	77	80
	5	0	0	0	0	0

Now we convert this profit matrix into loss matrix by subtracting all the elements from the highest element 111.

		Loss Matrix				
		A	B	C	D	E
Mechanics	1	49	33	61	0	29
	2	40	27	50	38	52
	3	24	19	0	40	30
	4	63	47	24	34	31
	5	111	111	111	111	111

We subtract the smallest element from all the elements in the respective rows.

1	49	33	61	0	29
2	13	0	23	11	25
3	24	19	0	40	30
4	39	23	0	10	7
5	0	0	0	0	0

Since each column has minimum element as zero, we draw minimum number of lines to cover all zeros.

	A	B	C	D	E
1	49	33	61	0	29
2	13	0	23	11	25
3	24	19	0	40	30
4	39	23	0	10	7
5	0	0	0	0	0

Here the number of lines drawn to cover all zeros =  $N = 4$ , which is less than the order of matrix.

We form the 2nd modified matrix by subtracting the smallest uncovered element from the remaining uncovered elements and adding to the element that is at the point of intersection of lines.

	A	B	C	D	E
1	49	40	68	0	29
2	6	0	23	11	18
3	17	19	0	33	23
4	32	23	0	3	0
5	0	7	7	0	0

Here,  $N = 5 = n = 5 =$  Order of matrix.

We make the assignment.

	A	B	C	D	E
1	49	40	68	0	29
2	6	0	23	4	18
3	17	19	0	33	23
4	32	23	X	3	0
5	0	7	7	X	X

The optimum assignment is

Jobs	Mechanics
A	5
B	2
C	3
D	1
E	4

Since the 5th mechanic is a dummy, job A is assigned to the 5th mechanic, this job is declined.

The maximum profit is given by,  $84 + 111 + 111 + 80 = \text{Rs } 386$ .

Example 4.12: A marketing manager has 5 salesmen and there are 5 sales districts. Considering the capabilities of the salesmen and the nature of districts, the estimates made by the marketing manager for the sales per month (in 1,000 rupees) for each salesman in each district would be as follows.

NOTES

	A	B	C	D	E
1	32	38	40	28	40
2	40	24	28	21	36
3	41	27	33	30	37
4	22	38	41	36	36
5	29	33	40	35	39

Find the assignment of salesmen to the districts that will result in the maximum sales.

**Solution:** We are given the profit matrix. To maximize the profit, first we convert it into a loss matrix, which can be minimized. To convert it into loss matrix, we subtract all the elements from the highest element 41. Subtract the smallest element from all the elements in the respective rows and columns, to get the first modified matrix.

Loss Matrix

	A	B	C	D	E
1	9	3	1	13	1
2	1	17	13	20	5
3	0	14	8	11	4
4	19	3	0	5	5
5	12	8	1	6	2

	A	B	C	D	E
1	8	2	0	12	0
2	0	16	12	19	4
3	0	14	8	11	4
4	19	3	0	5	5
5	11	7	0	5	1

	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

We now draw minimum number of lines to cover all zeros.

	A	B	C	D	E
1	8	0	0	7	0
2	0	14	12	14	4
3	0	12	8	6	4
4	19	1	0	0	5
5	11	5	0	0	1

$N = 4 < n = 5$

We subtract the smallest uncovered element from the remaining uncovered elements and add to the elements at the point of intersection of lines, to get the second modified matrix.

	A	B	C	D	E
1	0	0	1	8	0
2	0	13	12	4	3
3	0	11	8	6	3
4	0	0	0	0	4
5	11	4	0	0	0

Again,  $N = 4 < n = 5$ , which is less than the order of matrix. Repeat the above step.

	A	B	C	D	E
1	12	0	1	8	0
2	0	10	9	1	0
3	0	8	5	3	0
4	12	0	0	0	4
5	14	4	0	0	0

$N = 5 = n = 5$ , i.e., it is equal to the order of matrix. Hence, we make the assignment.

Assignment

	A	B	C	D	E
1	12	0	1	8	<del>0</del>
2	0	10	9	1	<del>0</del>
3	<del>0</del>	8	5	3	0
4	12	<del>0</del>	0	<del>0</del>	4
5	14	4	<del>0</del>	0	<del>0</del>

Since no row or column has single zero, we get a multiple solution.

(i) The optimum assignment is:

$1 \rightarrow B, 2 \rightarrow A, 3 \rightarrow E, 4 \rightarrow C, 5 \rightarrow D.$

With maximum profit  $(38 + 40 + 37 + 41 + 35) = \text{Rs } 191$

(ii) The optimum assignment is:

$1 \rightarrow B, 2 \rightarrow A, 3 \rightarrow E, 4 \rightarrow D, 5 \rightarrow C.$

Maximum profit  $(38 + 40 + 37 + 36 + 40) = \text{Rs } 191$

4.6 SUMMARY

- The assignment problem is one of the fundamental combinatorial optimization problems. It helps to find a maximum weight identical in nature in a weighted bipartite graph. The assignment problem is also termed as a special case of transportation problem.
- The solution of an assignment problem can be arrived at by using the Hungarian method.
- To Solve an assignment problem, prepare a cost matrix. If the cost matrix is not a square matrix then add a dummy row (column) with zero cost element.

... the minimum element in each row from all the elements of the respective rows.

- Job sequencing is basically the planning of the jobs in sequential manner and is an essential part of any work. Without proper planning and scheduling one can not achieve the desired output and profit.
- For sequencing a job, generally the two techniques are used termed as Priority Rules and Johnson's Rules.
- Priority rules give the guidelines for properly sequencing the job, where as Johnson's rule is used to minimize the completion time for a set of jobs to be done on two different machines.
- Priority rules are used to get specific guidelines for job sequencing. The rules do not consider job setup cost and time while analysing processing times. In it job processing time and due dates are given importance because the due dates are fixed to give delivery in time to the customers.
- The priority rules are very useful for process-focussed amenities, for example health clinics, print shops and manufacturing industries. Hence, priority rules minimize the time for completing a job, sequences the jobs in the organization, checks if any job is late and maximizes resource utilization.
- In First Come First Serve (FCFS), the job to be processed first is the job that turned up first in the organization.
- In Earliest Due Date (EDD), the job to be processed first is the job that has earliest due date.
- In Shortest Processing Time (SPT), the job to be processed first and completed is the job that is shortest in nature; in other words the job can be processed in short time.
- In Longest Processing Time (LPT), the job to be processed first is the job that is very important or of high priority though it can take longer processing time.
- In Critical Ratio (CR), the job to be processed first is analysed on the basis of critical ratio, which is an index number calculated from time remaining until due date divided by the remaining work time.
- Johnson's rule is applied to minimize the completion time for a set of jobs that are to be processed on two different machines or at two consecutive work stations.
- An assignment problem is said to be unbalanced if the cost matrix is not a square matrix, i.e., the number of rows and the number of columns are not equal. To make it balanced, we add a dummy row or a dummy column with all the entries as zeroes.

#### 4.7 KEY TERMS

- **Assignment problem:** A problem of finding an assignment of jobs among persons so that total cost of performing all jobs is at minimum

- **Cost matrix:** An assignment problem can be stated in form of  $n \times n$  matrix  $[C_{ij}]$  with  $i$  rows and  $j$  columns of real numbers, here,  $C_{ij}$  is the cost when  $i$ th person is assigned  $j$ th job
- **Unbalanced assignment:** An assignment is unbalanced if the cost matrix is not a square matrix

#### 4.8 ANSWERS TO 'CHECK YOUR PROGRESS'

1. It is a problem of finding an assignment of jobs among persons so that total cost of performing all jobs is at minimum.
2. An assignment problem can be stated in form of  $n \times n$  matrix  $[C_{ij}]$  with  $i$  rows and  $j$  columns of real numbers. Here,  $C_{ij}$  is the cost when  $i$ th person is assigned  $j$ th job.
3. Hungarian method is used to solve assignment problem.
4. There are nine steps in the method of solution by Hungarian method.
5. If the cost matrix of the assignment problem is not a square matrix, the problem is said to be unbalanced. To make it balanced, extra row(s) or column(s) are added with zero elements.
6. If the final cost matrix contains more than a required number of zeroes at independent positions, then it indicates the presence of an alternate optimal solution.
7. The maximization assignment problem can be converted into minimization assignment problem by subtracting all the elements in the given profit matrix from the highest element in that matrix.
8. The given profit matrix can be converted into a loss matrix or minimization type by subtracting all the elements from the highest element of the given matrix. For this minimization problem apply steps of the Hungarian method to get an optimal assignment.

#### 4.9 QUESTIONS AND EXERCISES

##### Short-Answer Questions

1. What is a cost matrix?
2. How does a cost matrix help solve assignment problems?
3. Write the first three steps of the Hungarian procedure for solving an assignment problem.
4. While solving a problem using the Hungarian method, at what stage are you able to make an assignment? Explain in brief.
5. While solving a problem using the Hungarian method, what is the condition that indicates a unique optimal solution?
6. What do you mean by maximization in an assignment problem?

7. If 4 jobs are to be assigned to 5 machines, how will you proceed to solve this problem?
8. State the mathematical form of an assignment problem.

### Long-Answer Questions

1. Explain the assignment problem using a suitable example.
2. Explain the significance and applications of an assignment problem.
3. Explain the algorithm for solving an assignment problem.
4. How can you maximize an objective function in an assignment problem?
5. Write the steps for finding a solution using the Hungarian method.
6. Solve the following assignment problems:

<p>(i)</p> <table style="margin-left: 20px;"> <tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th></tr> <tr><th>I</th><td>1</td><td>4</td><td>6</td><td>3</td></tr> <tr><th>II</th><td>9</td><td>7</td><td>10</td><td>9</td></tr> <tr><th>III</th><td>4</td><td>5</td><td>11</td><td>7</td></tr> <tr><th>IV</th><td>8</td><td>7</td><td>8</td><td>5</td></tr> </table>		A	B	C	D	I	1	4	6	3	II	9	7	10	9	III	4	5	11	7	IV	8	7	8	5	<p>(ii)</p> <table style="margin-left: 20px;"> <tr><th colspan="5">Men</th></tr> <tr><th></th><th>A</th><th>B</th><th>C</th><th>D</th></tr> <tr><th>I</th><td>10</td><td>25</td><td>15</td><td>20</td></tr> <tr><th>II</th><td>15</td><td>30</td><td>5</td><td>15</td></tr> <tr><th>III</th><td>35</td><td>20</td><td>12</td><td>24</td></tr> <tr><th>IV</th><td>17</td><td>25</td><td>24</td><td>20</td></tr> </table>	Men						A	B	C	D	I	10	25	15	20	II	15	30	5	15	III	35	20	12	24	IV	17	25	24	20
	A	B	C	D																																																				
I	1	4	6	3																																																				
II	9	7	10	9																																																				
III	4	5	11	7																																																				
IV	8	7	8	5																																																				
Men																																																								
	A	B	C	D																																																				
I	10	25	15	20																																																				
II	15	30	5	15																																																				
III	35	20	12	24																																																				
IV	17	25	24	20																																																				

(iii)

Men					
	A	B	C	D	E
I	1	3	2	8	8
II	2	4	3	1	5
III	5	6	3	4	6
IV	3	1	4	2	2
V	1	5	6	5	4

[Ans. (i) I → A, II → C, III → B, IV → D, Min cost = ₹21.  
 (ii) I → A, II → C, III → B, IV → D, Min time = 55 hours  
 (iii) A → I, B → IV, C → III, D → II, E → V]

7. There are five jobs to be assigned, one each to 5 machines. The associated cost matrix is as follows.

Machines					
	1	2	3	4	5
A	11	17	8	16	20
B	9	7	12	6	15
C	13	16	15	12	16
D	21	24	17	28	26
E	14	10	12	11	15

How should the job be assigned to various machines?

[Ans. A → 1, B → 4, C → 5, D → 3, E → 2, Min. cost = ₹60]

8. A company is faced with the problem of assigning 4 machines to 6 different jobs (one machine to one job only). The profits are estimated as follows.

Machines					
	A	B	C	D	
1	3	6	2	6	
2	7	1	4	4	
3	3	8	5	8	
4	6	4	3	7	
5	5	2	4	3	
6	5	7	6	4	

Solve the problem to maximize the profit.

9. Determine the optimum assignment schedule for the following assignment problem. The cost matrix is given below.

Machines						
	1	2	3	4	5	6
A	11	17	8	16	20	15
B	9	7	12	6	15	13
C	13	16	15	12	16	8
D	21	24	17	28	2	15
E	14	10	12	11	15	6

If the job C cannot be assigned to machine 6, will the optimum solution change?  
 [Ans. A → 3, B → 4, C → 1, D → 6, E → 2. Min. cost = ₹52]

## 4.10 FURTHER READING

- Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.
- Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research. Models and Methods*. New York: John Wiley & Sons.
- Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.
- Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.
- Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.

# UNIT 5 NETWORK ANALYSIS

## NOTES

### Structure

- 5.0 Introduction
- 5.1 Unit Objectives
- 5.2 Introduction to Network Concept
  - 5.2.1 Development of Network Analysis - CPM and PERT
- 5.3 Network Analysis and Rules of Network Construction
  - 5.3.1 Rules of Network Construction
  - 5.3.2 Time Analysis
  - 5.3.3 Network Diagram
- 5.4 Critical Path Method (CPM)
  - 5.4.1 Computations for Critical Path
  - 5.4.2 Applications of CPM Analysis
- 5.5 Programme Evaluation and Review Technique (PERT)
  - 5.5.1 PERT Procedure
- 5.6 Comparison and Limitations of PERT and CPM
- 5.7 Basic Concepts of Crashing
- 5.8 Resource Leveling and Resource Smoothing
- 5.9 Summary
- 5.10 Key Terms
- 5.11 Answers to 'Check Your Progress'
- 5.12 Questions and Exercises
- 5.13 Further Reading

## 5.0 INTRODUCTION

In this unit, you will learn about the various aspects of network analysis such as network scheduling, planning and control. Network scheduling is a technique used for planning and scheduling large projects in the field of construction, maintenance, fabrication, etc. It is a tool for minimizing problems in the execution and controlling critical factors in a project. Program Evaluation Review Technique (PERT) and Critical Path Method (CPM) are two planning and control techniques for keeping a project schedule on track to complete within the scheduled time.

You will learn the basic terms associated with network analysis technique. A network is a graphic representation of logically and sequentially connected arrows and nodes, representing the activities and events, respectively of a project. An event is the beginning and end points of an activity and is represented by a node. You will learn how to construct a network after going through the rules for network construction.

You will learn how to carry out the time analysis of a network model from its estimated completion time, earliest start time, earliest finishing time, latest start time and latest finishing time. You will learn how to analyse the critical path of a project to complete it within scheduled time.

## 5.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Know how to do network analysis for large projects
- Understand PERT and CPM for handling projects

- Construct a network of activities and events for analysis
- Understand the importance of CPM and PERT analysis
- Explain the PERT procedure
- Compare PERT and CPM
- Understand what crashing is
- Define resource levelling and resource smoothing

## NOTES

## 5.2 INTRODUCTION TO NETWORK CONCEPT

### Meaning and Objectives of Network Analysis

Network analysis is a method of planning and controlling projects by recording their interdependence in a diagrammatic form that enables each fundamental problem involved to be tackled separately. The main objectives of network analysis are:

1. To foster increased orderliness and consistency in the planning and evaluating of all areas in the project
  2. To provide an automatic mechanism for the identification of potential trouble spots in all areas which arise as a result of a failure in one.
  3. To structure a method to give operational flexibility to the programme by allowing for experimentation in a simulated sense.
  4. To effect speedy handling and analysis of the integrated data, thus allowing for expeditious correction of recognised trouble areas in project management.
- Network analyses, thus, plays an important role in project management. Through network analysis, which is a graphic depiction of 'activities' and 'events' related to a project, planning, scheduling and control of project becomes easier and effective.

### Steps Involved in Network Analysis

Network analyses achieve their purpose in three broad steps:

1. They present in diagrammatic form, a picture of all the jobs (or activities) to be accomplished and of their dependence on one another. The way in which this is done is to construct what is known as a 'network diagram' in which each job is represented by an arrow on the diagram. The way in which the arrows are linked indicates the dependencies of the jobs on each other.
2. They consider the limitations imposed by the availability of resources viz., of men, machine, money and material and in view of these estimate the time required to do each job.
3. They apply the estimated job time to the network diagram and then analyse the network. Analysis in this case means the calculation of the total length of time involved in each path through the network.

### Significance of Network Analysis

1. Network analysis clearly shows the interdependences between jobs to be performed in contest of a project and thus enables people to see not only the overall plan but the ways in which their own activities depend upon or influence those of others.
2. By splitting up the project into smaller activities, it assist in the estimation of their durations and thereby leading to more accurate target dates.
3. It enables stricter controls to be applied since any deviation from schedule is quickly noticed.
4. It allows the total requirements of men, materials, money, machinery and space resources to be readily calculated and also indicates where the delaying of non-critical jobs (i.e., jobs which do not immediately affect the duration of the project) may be used for optimal utilisation of resources.
5. Its identification of the critical path has two advantages: (i) If the completion date has to be advanced, attention can be concentrated on speeding up the relatively few 'critical' jobs, (ii) Money is not wasted on speeding up 'non critical' jobs.
6. It allows modifications of policy to be built easily and their impact can also be assessed quickly.
7. It allows schedules to be based on considerations of costs so as to complete projects in a given time at minimum expense.
8. It separates the planning of the sequence of jobs from the scheduling of times for the jobs and thus it makes planning and scheduling effective.

### Limitations of Network Analysis

The only real disadvantage of network analysis as a planning tool is that it is a tedious and exacting task if attempted manually. The calculations are done in terms of the sequence of activities and, if this is all that is required, a project involving several hundred activities may be attempted manually. However, the possibility of error is high, and if the results are to be sorted, the cost of manual operation rapidly becomes uneconomic. The consideration of various alternative plans also becomes impossible, because of the large volumes involved.

But now we have standard computer programmes for network analysis, which can handle project plans of upto 5000 activities and more, and produce 'output' in various forms. Even then it must be emphasised, that a computer only assists with the calculation and with the printing of *pirms* of operation sorted into various orders. The project manager is still responsible for the planning and must still make necessary decisions based upon the information supplied by the computer. The computer can not take over this responsibility. Equally important is the fact that the computer output is only as accurate as its input which is supplied in the first instance by human beings.

#### 5.2.1 Development of Network Analysis - CPM and PERT

Network analysis, as stated above, is a technique related to sequencing problems which are concerned with minimizing some measure or performance of the system

such as the total completion time of the project, the overall cost and so on. The technique is useful for describing the elements in a complex situation for the purpose of designing, planning, coordinating controlling and making decision. Network analysis is specially suited for projects which are not routine or repetitive and which will be conducted only once or a few times. Two most popular forms of this technique now used in many scheduling situations are the Critical Path Method (or simply CPM) and the Programme Evaluation and Review Technique (popularly known as PERT).

*Critical Path Method* or CPM was developed in 1956 at the E.I. du Pont de Nemours & Co., USA, to aid in the scheduling of routine plant overhaul maintenance and construction work. This method differentiates between planning and scheduling. Planning refers to the determination of activities that must be accomplished and the order in which such activities should be performed to achieve the objective of the project. Scheduling refers to the introduction of time into the plan thereby creating a time table for the various activities to be performed. CPM uses two time and two cost estimates for each activity (one time-cost estimate for the normal situation and the other estimate for the crash situation) but does not incorporate any statistical analysis in determining such time estimates. CPM operates on the assumption that there is a precise known time that each activity in the project will take.

*Programme Evaluation and Review Technique* or PERT was first developed in 1958 for use in defence projects specifically in the development of Polaris fleet ballistic missile programme. But now this technique is very popular in the hands of project planner and controller. PERT, now assists a business manager in planning and controlling a project. It allows a manager to calculate the expected total amount of time that the entire project will take to complete at the stage of formulation and planning a project and at the same time highlights the critical or the bottleneck activities in the project so that a manager may either allocate more resources for them or keep a careful watch on such activities as the project progresses. In PERT, we usually assume that the time to perform each activity is uncertain and as such three time estimates (the optimistic, the pessimistic and the most likely) are used. PERT is often described as an approach of multiple time estimates to scheduling problems of long-range research and development projects. PERT incorporates the statistical analysis in determining time estimates and enables the determination of the probabilities concerning the time by which each activity as well as the entire project would be completed. As such it can be taken as an advancement over the CPM. PERT is equally unique as a control device for it assists the management in controlling a project, once it has begun, by calling attention as a result of constant review to such delays in activities which might cause a delay in the project's completion date.

### 5.3 NETWORK ANALYSIS AND RULES OF NETWORK CONSTRUCTION

Network scheduling is a technique used for planning, and scheduling large projects in the field of construction, maintenance, fabrication, purchasing computer system, etc. The technique is a method of minimizing the trouble spots such as production, delays and interruptions, by determining critical factors and coordinating various parts of the overall job.

There are two basic planning and control technique that utilize a network to complete a predetermined project or schedule. These are Programme Evaluation Review Technique (PERT) and Critical Path Method (CPM).

A project is defined as a combination of interrelated activities all of which must be executed in a certain order for its completion.

The work involved in a project can be divided into three phases corresponding to the management functions of planning, scheduling and control.

**Planning:** This phase involves setting the objectives of the project and the assumptions to be made. Also it involves the listing of tasks or jobs that must be performed to complete a project under consideration. In this phase, men, machines and materials required for the project, in addition to the estimates of costs and duration of the various activities of the project, are also determined.

**Scheduling:** This consists of laying the activities according to the precedence order and determining,

- (i) The start and finish times for each activity.
- (ii) The critical path on which the activities require special attention.
- (iii) The slack and float for the non-critical paths.

**Controlling:** This phase is exercised after the planning and scheduling, which involves the following:

- (i) Making periodical progress reports.
- (ii) Reviewing the progress.
- (iii) Analysing the status of the project.
- (iv) Management decisions regarding updating, crashing and resource allocation.

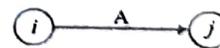
#### Basic Terms

To understand the network techniques one should be familiar with few basic terms of which both CPM and PERT are special applications.

**Network:** It is the graphic representation of logically and sequentially connected arrows and nodes representing activities and events of a project. Networks are also called arrow diagram.

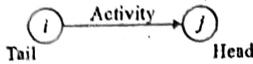
**Activity:** An activity represents some action and is a time consuming effort necessary to complete a particular part of the overall project. Thus, each and every activity has a point of time where it begins and a point where it ends.

It is represented in the network by an arrow,

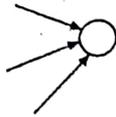


Here, A is called the activity.

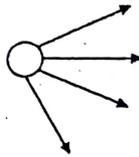
**Event:** The beginning and end points of an activity are called events or nodes. Event is a point in the time and does not consume any resource. It is represented by a numbered circle. The head event called the  $j$ th event has always a number higher than the tail event called the  $i$ th event.



**Merge and Burst Events:** It is not necessary for an event to be the ending event of only one activity but can be the ending event of two or more activities. Such event is defined as a merge event.



If the event happens to be the beginning event of two or more activities it is defined as a burst event.



**Preceding, Succeeding and Concurrent Activities:** Activities, which must be accomplished before a given event can occur are termed as *preceding activities*.

Activities, which cannot be accomplished until an event has occurred are termed as *succeeding activities*.

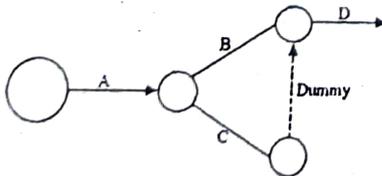
Activities, which can be accomplished concurrently are known as *concurrent activities*.

This classification is relative, which means that one activity can be preceding to a certain event, and the same activity can be succeeding to some other event or it may be a concurrent activity with one or more activities.

**Dummy Activity:** Certain activities, which neither consumes time nor resources but are used simply to represent a connection or a link between the events are known as dummies. It is shown in the network by a dotted line. The purpose of introducing dummy activity is as follows:

(i) To maintain uniqueness in the numbering system as every activity may have distinct set of events by which the activity can be identified.

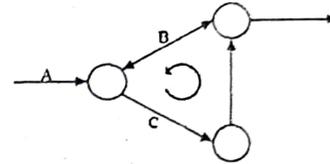
(ii) To maintain a proper logic in the network.



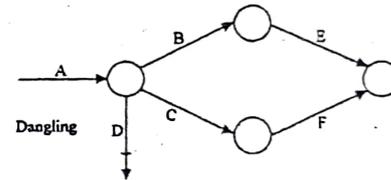
## Common Errors

Following are the three common errors in a network construction:

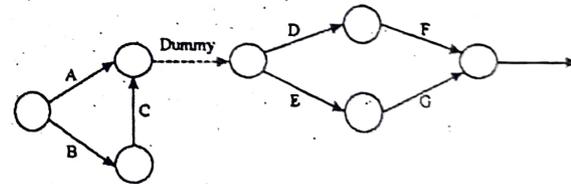
**Looping (Cycling):** In a network diagram looping error is also known as cycling error. Drawing an endless loop in a network is known as error of looping. A loop can be formed if an activity were represented as going back in time.



**Dangling:** To disconnect an activity before the completion of all the activities in a network diagram is known as dangling.



**Redundancy:** If a dummy activity is the only activity emanating from an event and which can be eliminated is known as redundancy.



## 5.3.1 Rules of Network Construction

There are a number of rules in connection with the handling of events and activities of a project network that should be followed.

- (i) Try to avoid arrows which cross each other.
- (ii) Use straight arrows.
- (iii) No event can occur until every activity preceding it has been completed.
- (iv) An event cannot occur twice, i.e., there must be no loops.
- (v) An activity succeeding an event cannot be started until that event has occurred.
- (vi) Use arrows from left to right. Avoid mixing two directions. Vertical and standing arrows may be used if necessary.
- (vii) Dummies should be introduced if it is extremely necessary.
- (viii) The network has only one entry point called the start event and one point of emergence called the end or terminal event.

### Numbering the Events (Fulkerson's Rule)

After the network is drawn in a logical sequence, every event is assigned a number. The number sequence must reflect the flow of the network. In numbering the events the following rules should be observed:

- (i) Event numbers should be unique.
- (ii) Event numbering should be carried out on a sequential basis from left to right.
- (iii) The initial event which has all outgoing arrows with no incoming arrow is numbered as 1.
- (iv) Delete all arrows emerging from all the numbered events. This will create at least one new start event out of the preceding events.
- (v) Number all new start events 2, 3 and so on. Repeat this process until all the terminal event without any successor activity is reached. Number the terminal node suitably.

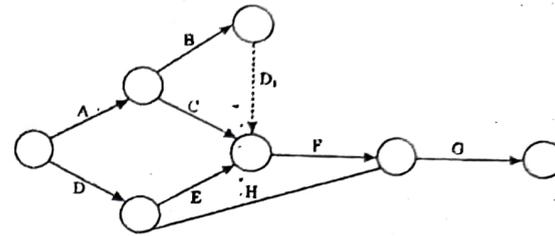
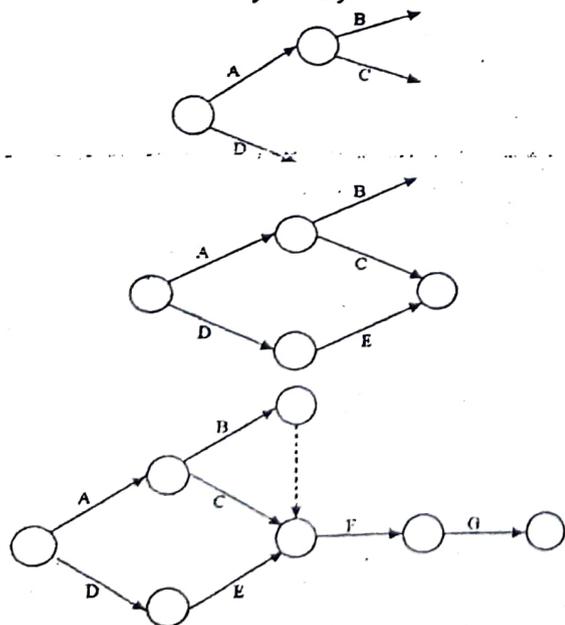
*Note:* The head of an arrow should always bear a number higher than the one assigned to the tail of the arrow.

### Construction of Network

**Example 5.1:** Construct a network for the project whose activities and their precedence relationships are as given below:

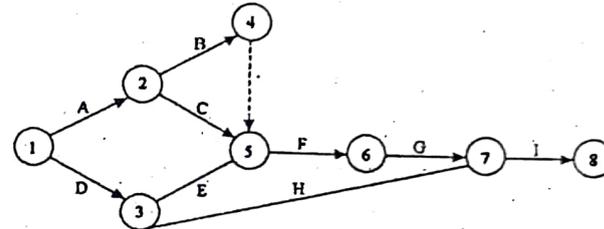
Activities	A	B	C	D	E	F	G	H	I
Immediate Predecessor	-	A	A	-	D	B,C,E	F	D	G,H

**Solution:** From the given constraints, it is clear that *A, D* are the starting activity and *I* the terminal activity. *B, C* are starting with the same event and are both the predecessors of the activity *F*. Also *E* has to be the predecessor of both *F* and *H*. Hence, we have to introduce a dummy activity.



$D_1$  is the dummy activity.

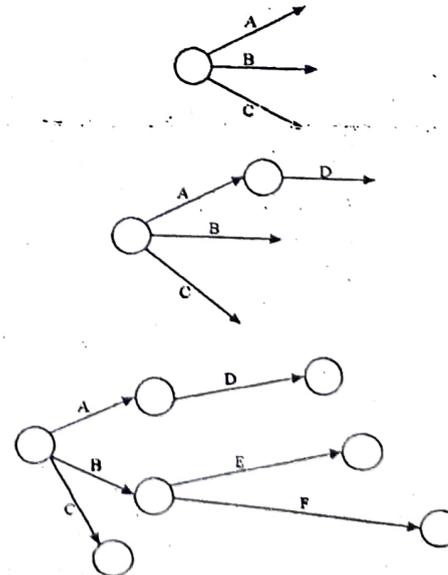
Finally we have the following network:

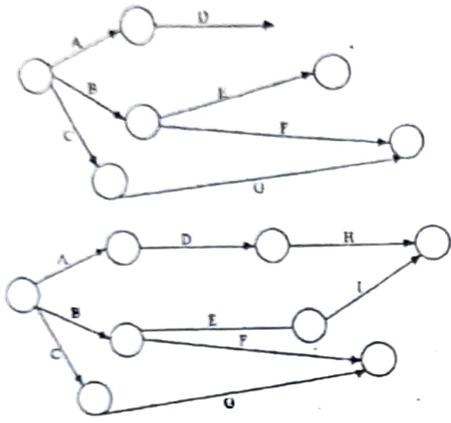


**Example 5.2:** Construct a network for each of the projects whose activities and their precedence relationships are given below.

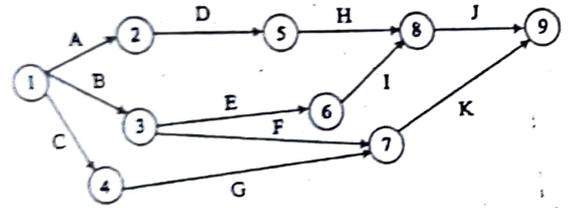
Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	-	-	-	A	B	B	C	D	E	H,I	F,G

**Solution:** *A, B, C* are the concurrent activities as they start simultaneously. *B* becomes the predecessor of activity *E* and *F*. Since the activities *J, K* have two preceding activities, dummy may be introduced (if possible).





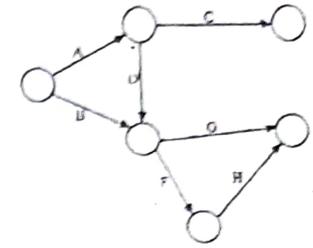
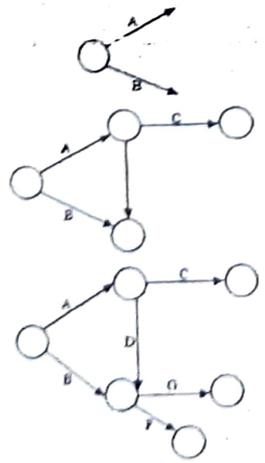
Finally we have,



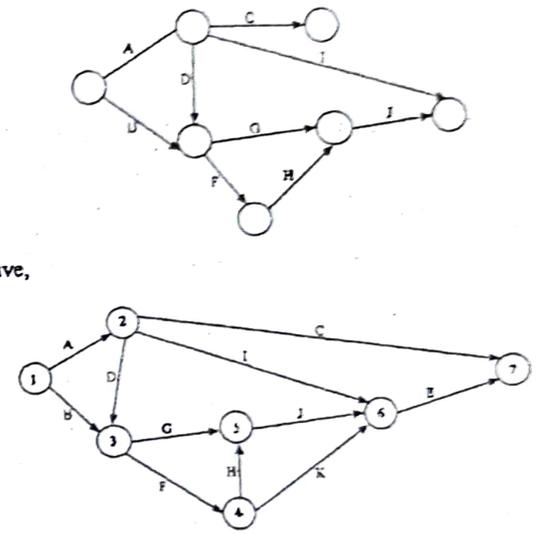
**Example 5.3:** Construct a network of the project whose activities are given below.  
 $A < C, D, I, B < G, F, D < G, F, F < H, K; G, H < J; I, J, K < E$   
**Solution:** Given  $A < C$  which means that  $C$  cannot be started until  $A$  is completed, i.e.,  $A$  is the preceding activity to  $C$ . The above constraints can be given in the following table:

Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor	-	-	A	A	I, J, K	B, D	B, D	F	A	G, H	F

$A, B$  are the starting activity, and  $E$  is the terminal activity.



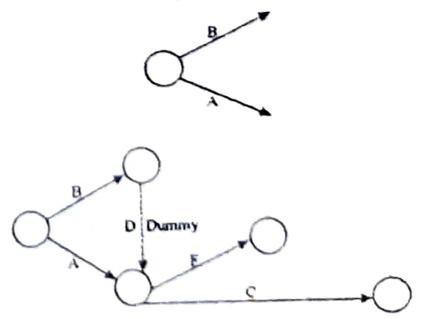
Finally we have,

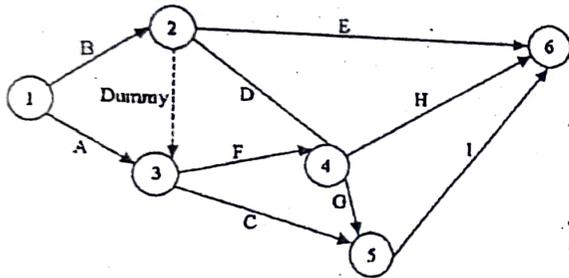
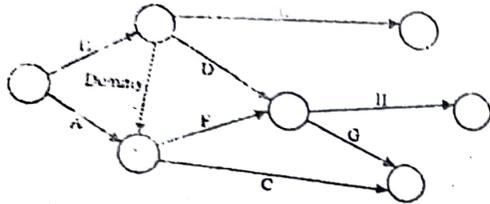
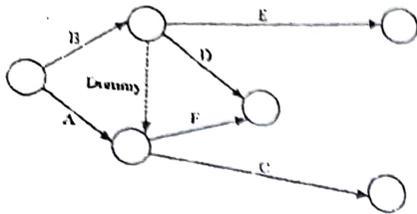


**Example 5.4:** Construct the network for the project whose activities and precedence relationship is given below. Show also the dummy activity.

Activities	A	B	C	D	E	F	G	H	I
Immediate Predecessor	-	-	A, B	B	B	A, B	F, D	F, D	C, G

**Solution:**  $A, B$  are concurrent activities as they start simultaneously.  $I$  is the terminal activity. Since the activities  $C$  and  $F$  are coming from both the activities  $A, B$  we need to introduce a dummy activity.





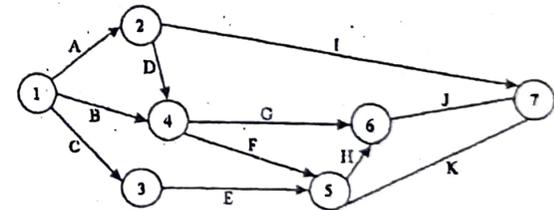
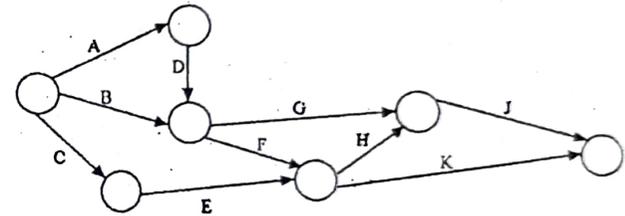
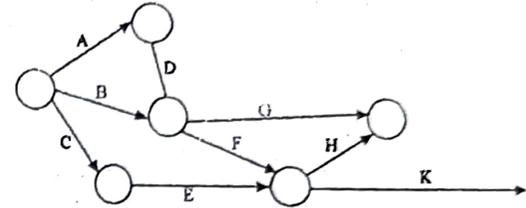
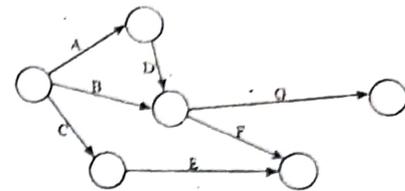
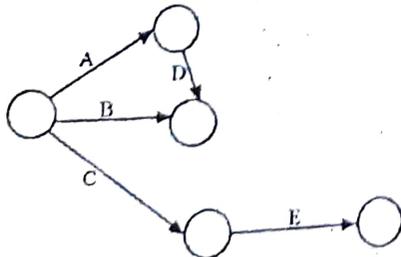
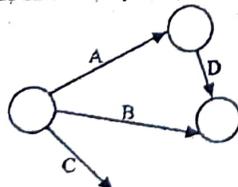
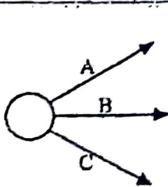
Example 5.5: Make a network of the project having activities and precedence relationship as given below:

A, B, C can start simultaneously,

$A < D, I; B < G, F; D < G, F; C < E; E < H, K; F < H, K; G, H < J$

Solution: The above constraints can be formatted into a table.

Activity	A	B	C	D	E	F	G	H	I	J	K
Predecessor Activity	-	-	-	A	C	B, D	B, D	E, F	A	G, H	E, F



### 5.3.2 Time Analysis

Once the network of a project is constructed the time analysis of the network becomes essential for planning various activities of the project. An activity time is a forecast of the time an activity is expected to take from its starting point to its completion (under normal conditions).

We shall use the following notation for basic scheduling computations.

$(i, j)$  = Activity  $(i, j)$  with tail event  $i$  and head event  $j$

$t_{ij}$  = Estimated completion time of activity  $(i, j)$

$(ES)_{ij}$  = Earliest starting time of activity  $(i, j)$

$(EF)_{ij}$  = Earliest finishing time of activity  $(i, j)$

$(LS)_{ij}$  = Latest starting time of activity  $(i, j)$

$(LF)_{ij}$  = Latest finishing time of activity  $(i, j)$

The basic scheduling computation can be put under the following three groups.

## NOTES

**Forward Pass Computations (For Earliest Event Time)**

Before starting computations, the occurrence time of the initial network event is fixed. The forward pass computation yields the earliest start and the earliest finish time for each activity  $(i, j)$  and indirectly the earliest occurrence time for each event namely  $E_i$ . This consists of the following three steps:

**Step 1:** The computations begin from the start node and move towards the end node. Let zero be the starting time for the project.

**Step 2:** Earliest starting time  $(ES)_i = E_i$  is the earliest possible time when an activity can begin assuming that all of the predecessors are also started at their earliest starting time. Earliest finish time of activity  $(i, j)$  is the, Earliest starting time + Activity time

$$(EF)_j = (ES)_i + t_{ij}$$

**Step 3:** Earliest event time for event  $j$  is the maximum of the earliest finish time of all the activities ending at that event.

$$E_j = \text{Max}_i (E_i + t_{ij})$$

The computed 'E' values are put over the respective rectangle  representing each event.

**Backward Pass Computations (For Latest Allowable Time)**

The latest event time (L) indicates the time by which all activities entering into that event must be completed without delaying the completion of the project. These can be calculated by reversing the method of calculations used for the earliest event time. This is done in the following steps:

**Step 1:** For ending event assume  $E = L$ .

**Step 2:** Latest finish time for activity  $(i, j)$  is the target time for completing the project

$$(LF)_j = L_j$$

**Step 3:** Latest starting time of the activity  $(i, j) =$  Latest completion time of  $(i, j)$ , the activity time.

$$(LS)_i = (LF)_j - t_{ij} \\ = L_j - t_{ij}$$

**Step 4:** Latest event time for event  $i$  is the minimum of the latest start time of all activities originating from the event.

$$L_i = \text{Min}_j (L_j - t_{ij})$$

The computed 'L' values are put over the respective triangle  $\Delta$  representing each event.

**Determination of Floats and Slack Times**

Float is defined as the difference between the latest and the earliest activity time.

Slack is defined as the difference between the latest and the earliest event time.

Hence, the basic difference between the slack and the float is that slack is used for events only whereas float is used for activities.

There are mainly three kinds of floats as given below:

**Total Float:** It refers to the amount of time by which the completion of an activity could be delayed beyond the earliest expected completion time without affecting the overall project duration time.

Mathematically, the Total Float (TF) of an activity  $(i, j)$  is the difference between the latest start time and the earliest start time of that activity.

Hence, the total float for an activity  $(i, j)$  denoted by  $(TF)_{ij}$  is calculated by the formula,

$$(TF)_{ij} = (\text{Latest start} - \text{Earliest start}) \text{ for activity } (i, j)$$

$$\text{i.e., } (TF)_{ij} = (LS)_{ij} - (ES)_{ij}$$

$$\text{or, } (TF)_{ij} = (L_j - E_i) - t_{ij}$$

Where  $E_i, L_j$  are the earliest time and latest time for the tail event  $i$  and head event  $j$  and  $t_{ij}$  is the normal time for the activity  $(i, j)$ . This is the most important type of float as it concerns with the overall project duration.

**Free Float:** The time by which the completion of an activity can be delayed beyond the earliest finish time without affecting the earliest start of a subsequent succeeding activity.

Mathematically, the Free Float for activity  $(i, j)$  denoted by  $(FF)_{ij}$  can be calculated by the formula,

$$FF_{ij} = (E_j - E_i) - t_{ij}$$

$$FF_{ij} = \text{Total float} - \text{Head event slack}$$

$$\text{Head event slack} = L_j - E_j$$

This float is concerned with the commencement of subsequent activity.

The free float can take values from zero up to total float, but it cannot exceed total float. This float is very useful for rescheduling the activities with minimum disruption of earlier plans.

**Independent Float:** The amount of time by which the start of an activity can be delayed without affecting the earliest start time of any immediately following activities assuming that the preceding activity has finished at its latest finish time.

Mathematically, Independent Float of an activity  $(i, j)$  denoted by  $(IF)_{ij}$  can be calculated by the formula,

$$(IF)_{ij} = (E_j - L_i) - t_{ij}$$

or

$$(IF)_{ji} = \text{Free float} - \text{Tail event slack}$$

Where tail event slack is given by,

$$\text{Tail event slack} = L_i - E_i$$

... independent float is always taken as zero. This float is concerned with prior and subsequent activities.

$$(IF)_y \leq (FF)_y \leq (TF)_y$$

Notes:

1. If the total float  $TF_y$  for any activity  $(i, j)$  is zero, then those activities are called *critical activity*.
2. The float can be used to reduce project duration. While doing this, the float of not only that activity but that of other activities would also change.

**Critical Activity:** An activity is said to be critical if a delay in its start will cause a further delay in the completion of the entire project.

**Critical Path:** The sequence of critical activities in a network is called the *critical path*. It is the longest path in the network from the starting event to the ending event and defines the minimum time required to complete the project. In the network, it is denoted by double line. This path identifies all the critical activities of the project. Hence, for the activity  $(i, j)$  to lie on the critical path, following conditions must be satisfied.

(i)  $ES_i = LF_i$

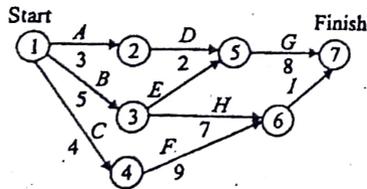
(ii)  $ES_j = LF_j$

(iii)  $ES_j - ES_i = LF_j - LF_i = t_{ij}$

$ES_i, ES_j$  are the earliest start and finish time of the event  $i$  and  $j$ .

$LF_i, LF_j$  are the latest start, finish time of the event  $i$  and  $j$ .

### 5.3.3 Network Diagram



In the above diagram, each arrow represents an activity and each circle an event. Circle 1 represents the starting event and circle 7 represents the ending event. The names of the activities are generally stated just above the corresponding arrows. Thus A in the above diagram is the name of the activity represented by the arrow just drawn below it.

- **Merge and Burst events:** It may be pointed out that it is not necessary for an event to be the ending event of only one activity but an event can be the ending event of two or more activities in which case the said event is technically described as *merge event*. Similarly, if the event happens to be the beginning event of two or more activities it is technically called as the *'burst event'*.
- **Preceding, Succeeding and Concurrent activities:** The activities can be classified as preceding activities; succeeding activities and the concurrent activities. Activities which must be accomplished before a given event can occur are termed as *preceding activities*; activities which cannot be accomplished until an event has occurred are termed as *succeeding activities* and activities which can be accomplished concurrently are known as *concurrent*

*activities*. This classification is relative which means that one activity can be preceding to a certain event and the same activity can be succeeding to some other event or it may be a concurrent activity with one or more of the activities.

- **Dummy activities:** Some times we use dummy activities in the preparation of network diagram. Such activities are to designate a precedence relationship and in the network diagram are shown as broken lines. They are characterized by their use of zero time and zero resource. Their main function is to help in assuring that the activities and events in a network diagram are in proper sequence.
- **Path and Critical path:** A path is continuous chain of activities through a network which connects the first event to the last event. Critical path consists of the sequence of those events and connected activities that require the maximum time in the completion of the project. It is that path which takes the longest time. It is known as critical because it controls the completion date of the project. The length of this path determines the minimum time in which the project may be completed.
- **Critical activities or Bottleneck activities:** All the activities associated with the critical path are called as *critical* or *bottleneck activities*. Any delay in the completion of one or more of these activities will cause delay in the completion date of the project. Hence such activities require special attention of the project incharge.
- **Earlier start time or Est:** Est for an activity is the earliest possible time an activity can begin on the assumption that all activities preceding to it started at the earliest possible times.
- **Earliest finish time or Eft:** Eft is the sum of the earliest start time and the estimated time to perform the concerning activity.
- **Latest finish time or Lft:** Lft for an activity is the latest possible time an activity can finish without delaying the project beyond its dead line on the assumption that all the subsequent activities are performed as planned.
- **Latest start time (or Lst):** Lst for an activity is the difference between the latest finish time and the estimated time for the activity to be performed.
- **Float (Total, Interfering Independent and Free floats):** Quite often the term float (in CPM terminology) is used in context of network analysis. Float may be understood as total float, interfering float, free float and independent float. *Total float* is the duration by which an activity can be delayed without delaying the project and can be worked out as either  $(Lst-Est)$  or  $(Lft-Eft)$ . *Interfering float* is that part of the total float which causes a reduction in the float of the successor activity or activities. In other words, it is that portion of the activity float which cannot be consumed without affecting adversely the float of the succeeding activity or activities. It is worked out as a difference between the *Lft* of the activity and the *Est* of the following activity or zero whichever is larger. *Interfering float* is also known as the head event slack of an activity. *Free float* is that portion of the total float within which an activity can be manipulated without affecting the float of subsequent activities. It is worked out by subtracting the head event slack from the total float. The head event slack is its latest event time minus earliest

NOTES

event time or  $(L_T - E_T)$ . *Independent float* is that portion of the total float within which an activity can be delayed for started without affecting float of the preceding activities. It is worked out by subtracting the tail event slack from the free float. If it obtains a negative value then it is taken as equal to zero. *Tail event* refers to the event where an activity say begins and *Head event* is the event where an activity comes to an end. If we have events (1) and (2) then (1) is the tail event and (2) is the head event of an activity A.

Float may be positive or negative. Positive float indicates that the activities concerned have certain amount of spare time and can be delayed without effecting the project duration. On the other hand, negative float highlights the situation in which the activities concerned are short of time and unless their duration (to the extent of negative float) is reduced, completion of the project by the target time cannot be assured. Thus, negative float indicates the extent of criticality of the activities.

- *Slack*: The term slack is normally associated with events. It indicates the amount of latitude that is available for an event to occur. It is worked out as under:

Slack of an event = (Latest occurrence time of the event) - (Earlier occurrence time of the event) or simply slack of event =  $(L_T - E_T)$ . Slack can be positive or negative depending upon whether the targetted date of completion is later or earlier than the earliest finish time of the task respectively.

When used for activities, the term slack should be used for activity slack (activity slack is synonymous to float). Since slack is associated with the events, each activity will have two slacks which includes the slack of its head event or the head slack and the slack of its tail event or the tail slack.

**Preparation of the Network Arrow Diagram**

We require the following information for each activity in the project for the preparation of the network diagram:

- The sequencing requirements for an activity must be known, i.e., the set of activities which must be completed prior to the beginning of each specific activity should be known.
- An estimate of the time each activity will take should also be known.

Keeping all what has been stated above in view, the network diagram can easily be prepared. But the following rules of constructing network diagrams will have to be invariably adhered:

- Each activity is shown by an arrow only once in the network.
- Network has to be developed on the basis of logical dependencies between various activities.
- The length of arrows representing various activities have no significance; they only indicate the logical precedence.
- Arrow direction shows the general progression in time.
- Events in the network are shown by numbers.
- Activities are identified by the numbers of their starting and the ending events.

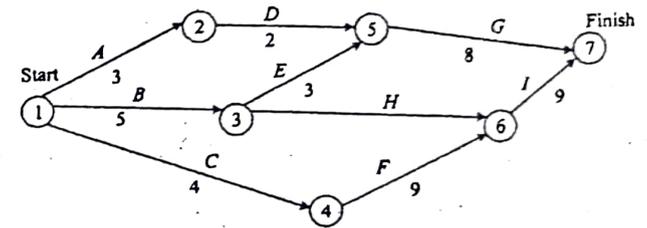
- Parallel activities between two events without intervening events are not permitted. In such a situation dummy activities may have to be introduced.
- Looping is not permitted in a network. This means that if activity A precedes B and B precedes C, then C cannot precede A.

Now construct the network diagrams using the above stated rules.

**Examples 5.6:** Prepare a network arrow diagram for the following information.

Activity	Name of the Activity	Pre-requisite Activity	Estimated Time (Weeks)
Event 1 to Event 2	A	None	3
Event 1 to Event 3	B	None	5
Event 1 to Event 4	C	None	4
Event 2 to Event 5	D	None	2
Event 3 to Event 5	E	A	3
Event 4 to Event 6	F	B	9
Event 5 to Event 7	G	C	8
Event 3 to Event 6	H	D	7
Event 6 to Event 7	I	E	9

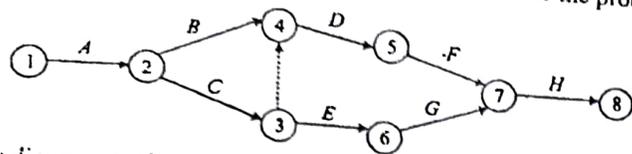
**Solution:** Draw the following network arrow diagram to solve the problem:



The above is the required network diagram for the given problem. Activity A must be completed before activity D can begin the arrow activities. The immediate preceding activity of activity D is activity A which means that activity A must be completed before activity D can begin. The arrow from circle 1 to circle 2 indicate that activity A must be completed before activity D can begin. Similarly activity B must be completed before activities E and/or H can. C must be completed before F can begin; activities H and F must be completed before I can begin and activities D and E must be completed before activity G can begin. The estimated time for each activity has been placed just below the arrow representing that activity.

**Example 5.7:** Draw a network arrow diagram for the following information concerning some project:

Activity	Predecessor Activity or Activities
A	None
B	A
C	A
D	B, C
E	C
F	D
G	E
H	F, G



In this diagram, activity 3-4 is the dummy activity shown as a broken line. It is required because activities B and C both precede activity D but activity C alone precedes activity E.

## 5.4 CRITICAL PATH METHOD (CPM)

Critical path method is a graphical technique for planning and scheduling of projects. This technique involves the preparation of the network in the form of arrow diagram and its analysis to indicate the critical path. It has the potential for scheduling of a task in minimum time and/or cost in accordance with specified constraints.

After preparing the network diagram and indicating the times for each activity, we can now mention the various possible paths, for determining the critical path. The critical path being the longest path can easily be found out from the possible paths as the one taking the maximum time in the completion of the project. In the network diagram of Example 5.6 there are in all four paths viz.

1.  $A \rightarrow D \rightarrow G$  requiring  $3 + 2 + 8 = 13$  weeks in completion of the project.
2.  $B \rightarrow E \rightarrow G$  requiring  $5 + 3 + 8 = 16$  weeks in completion of the project.
3.  $B \rightarrow H \rightarrow I$  requiring  $5 + 7 + 9 = 21$  weeks in completion of the project.
4.  $C \rightarrow F \rightarrow I$  requiring  $4 + 9 + 9 = 22$  weeks in completion of the project.

As the path  $C \rightarrow F \rightarrow I$  takes longest time in the completion of the project, it is the critical path. The activities C, F and I are associated with it and hence they are critical or bottleneck activities. All other activities viz., A, B, D, E, G and H are non-critical activities. Non-critical activities have a certain amount of spare time or float available. These activities can be delayed to the extent of the float available without affecting the overall completion time of the project

The critical path is important as its length determines the minimum time required for the completion of the project. The critical path requires greater attention because of the following reasons:

- The critical path highlights those activities which must be performed more rapidly if the total project completion time is to be reduced;
- Any delay in activities which are on the critical path will produce delay in the completion of the project, i.e., will postpone the final completion date of the project on the other hand delays in non-critical activities may not actually delay the completion of the project;
- Advance planning and improvement along the critical path may cause another path to become critical.

In brief, the critical path directs management's attention to important facts, spots potential bottlenecks and avoids unnecessary pressure on other paths that will not result in an earlier final completion date of the project.

### 5.4.1 Computations for Critical Path

The computations to be accomplished for critical path are as follows: The Earliest Start Time (or Est) and the Earliest finish time (or Eft) for each activity are to be obtained first. For this purpose, we set the Est of the first activity equal to zero. Then add the estimated time to perform the first activity to its Est and the result is the Eft for the first activity. Now, take any activity for which all of its immediate preceding activities have Est and Eft values. The Est of such an activity is equal to the largest of the Eft values of its immediate preceding activities. If we proceed this way (i.e., from left to right), finding the Est and Eft of all activities in the network, we are said to adopt what is known as the *Forward Pass*.

Similarly, we will have to work out the Latest Start Time (or Lst) and the Latest Finish Time (or Lft) for each activity to be performed in the completion of the project. This can be done as under:

Start at the end of the network diagram and first set the Lft for the last activity equal to the Eft for that activity. Then subtract the estimated time to perform the last activity from its Lft to obtain its Lst. Now take any activity for which all of its immediate succeeding activities have Lst and Lft values. The Lft of such an activity is equal to the smallest of the Lst values of its immediate succeeding activities. If we proceed this way (i.e., from right to left) finding the Lst and Lft of all activities in the network, we are said to adopt what is known as the *Backward Pass*.

**Activity Floats:** Often the complete date of the project is determined from the length of the critical path, the float will be positive except along the critical path where it will be zero. The float for each activity can be calculated by taking the difference either between the Lst and the Est or between Lft and Eft for that activity. *Float will always be zero for those activities which are on the critical path.* This is a the definition of critical path since any delay in a critical activity will cause delay in the completion date of the project.

**'Slack' in case of events:** In case of events we usually talk of slack and for activities we can think of their head events slacks and tail events slacks.

Slack for events concerning illustration one can be shown as under:

For Event	$L_T$	$E_T$	Slack (i.e., $L_T - E_T$ )
1	6	0	0
2	12	3	9
3	6	5	1
4	4	4	0
5	14	8	6
6	13	13	0
7	22	22	0

$L_T$  = Latest Event Time

$E_T$  = Earliest Event Time

Head event slack tail event slack relating to an activity can be shown as under:

For Activity	Head event Slack	Tail event Slack
A	9	0
B	1	0
C	0	0
D	6	9
E	6	1
F	0	0
G	0	6
H	0	1
I	0	0

Float or slack whether positive or negative, is generally considered as undesirable and should be avoided to the extent possible. *Positive float* simply means that there is idle time and resources with corresponding simplicities costs burden. When the time to be allowed for the completion of the project is less than the time which is required as per the critical path analysis for the completion of the project, we have negative float which simply represents that project requires more resources than are normally available. In such a situation, the project manager can either choose not to meet the completion date and bear the burden of penalties, if any, that may be imposed or to use more resources, i.e., to work on the basis of crash plan and absorb the corresponding increase in costs in order to complete the project within the stipulated time. *Negative float* is a sort of warning that final event will not be completed on schedule with the existing plan. It thus serves to indicate the extent of criticality of the activity. Once we have determined the critical path and have worked out the floats in respect of each activity, then adjustments can be made for better utilization of resources and time. Some of the possible adjustments can be as under:

- (i) Reduction of time estimates of bottleneck activities.
- (ii) Eliminations of some activities, if possible.
- (iii) Bringing in some more resources.
- (iv) Transferring resources from activities having float to critical activities with zero float.
- (v) Restructuring of the network with a view to reduce completion time of the project.

With one or more of the above stated adjustments, the CPM analysis can result in producing an improved plan for the completion of the project in time and that too in an economical manner. Besides, when new information comes as the project progresses, the plan can be reevaluated and revised to incorporate the new developments. If used in this manner, the network analysis proves to be a dynamic device for effecting control over the project.

### Resource Allocation and Levelling

Resource Allocation (also known as resource scheduling) implies the task of allocation of resources to various activities in such a manner that the allocation is considered as acceptable under the given situation. The task of allocation of resources is of vital importance as the final schedule depends upon the quantity of deployment of resources. Then the basic question is: How the resources should be allocated? This depends upon several factors like availability of resources, requirements, restrictions in regard

to completion date, etc. Various types of problems may be encountered in this connection, but we shall consider only two of such problems: (i) Resource levelling and (ii) Limited resource allocation.

### The Problem of Resource Levelling

Resource Levelling (also known as local smoothing) means the resource scheduling exercise in which the resource demand is evened out or levelled as much as possible. In other words, resource levelling refers to the scheduling of activities within the limits of the available floats in such a way that variations in resource requirements are minimized. Though no constraint is put on the availability of resources in context of resource levelling but the aggregate demand of each of the important resources requires to be levelled up so as to minimize resource costs.

### The Problem of Resource Allocation

Problem arises when the resource/resources are limited. We quite often find that projects require costly items of plant and equipments for execution of the work of which only a limited number are available. Such limited resources must be allocated with a lot of care so that the total requirement should not exceed the ceiling and the utilization factor remains high. This necessitates rescheduling of some or all of the activities and may even involve delay in overall completion of the project.

The methodology for resource levelling involves the following steps:

- (i) Prepare the list of the resources that would be required for execution of the various activities.
- (ii) Prepare the resource profiles for each resource by resource aggregation exercise.
- (iii) Identify the periods of peak and low demands.
- (iv) Make an attempt to lower down the demand in peak periods to fill up the troughs, i.e., to make the demand as uniform as possible. This can be done by altering the times of start and finish of non-critical activities in accordance of their floats without affecting the overall completion date of the project.

### 5.4.2 Applications of CPM Analysis

The iterative procedure of determining the critical path is as follows.

**Step 1:** List all the jobs and then draw arrow (network) diagram. Each job is indicated by an arrow with the direction of the arrow showing the sequence of jobs. The length of the arrows has no significance. The arrows are placed based on the predecessor, successor, and concurrent relation within the job.

**Step 2:** Indicate the normal time ( $t_p$ ) for each activity ( $i, j$ ) above the arrow which is deterministic.

**Step 3:** Calculate the earliest start time and the earliest finish time for each event and write the earliest time  $E_i$  for each event  $i$  in the  $\square$ . Also calculate the latest finish and latest start time. From this we calculate the latest time  $L_j$  for each event  $j$  and put in the  $\Delta$ .

the arrow diagram.

Step 5: Determine the total float for each activity by taking the difference between the earliest start and the latest start time.

Step 6: Identify the critical activities and connect them with the beginning event and the ending event in the network diagram by double line arrows. Which gives the critical path.

Step 7: Calculate the total project duration.

Note: The earliest start, finish time of an activity, and the latest start, finish time of an activity are shown in the table. These are calculated by using the following hints.

To find the earliest time we consider the tail event of the activity. Let the starting time of the project, namely  $ES_1 = 0$ . Add the normal time with the starting time to get the earliest finish time. The earliest starting time for the tail event of the next activity is given by the maximum of the earliest finish time for the head event of the previous activity.

Similarly, to get the latest time, we consider the head event of the activity.

The latest finish time of the head event of the final activity is given by the target time of the project. The latest start time can be obtained by subtracting the normal time of that activity. The latest finish time for the head event of the next activity is given by the minimum of the latest start time for the tail event of the previous activity.

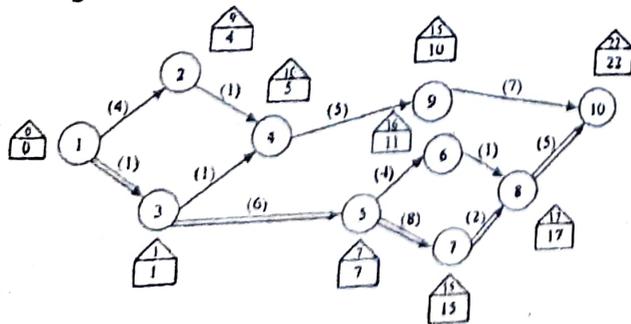
Example 5.8: A project schedule has the following characteristics.

Activity	1-2	1-3	2-4	3-4	3-5	4-9	5-6	5-7
Time(days)	4	1	1	1	6	5	4	8
Activity	6-8	7-8	8-10	9-10				
Time(days)	1	2	5	7				

From the above information, you are required to:

- Construct a network diagram.
- Compute the earliest event time and latest event time.
- Determine the critical path and total project duration.
- Compute total, free float for each activity.

Solution: First we construct the network with the given constraints. Here we get this by just connecting the event numbers.



The following table gives the critical path, total and, free floats calculation.

Activity	Normal Time	Earliest		Latest		TF	FF
		Start	Finish	Start	Finish		
1-2	4	0	4	5	9	5	5-5=0
1-3	1	0	1	0	1	0	0
2-4	1	4	5	9	10	5	0
3-4	1	1	2	9	10	8	3
3-5	6	1	7	1	7	0	0
4-9	5	5	10	10	15	5	0
5-6	4	7	11	12	16	5	0
5-7	8	7	15	7	15	0	0
6-8	1	11	12	16	17	5	5
7-8	2	15	17	15	17	0	0
8-10	5	17	22	17	22	0	0
9-10	7	10	17	15	22	5	5

The earliest and latest calculations are shown below:

Forward Pass Calculation: In this we estimate the earliest start  $ES$ , and finish time  $ES$ . The earliest time for the event  $i$  is given by,

$$E_i = \text{Max} (ES_j + t_{ij})$$

$$ES_1 = 0 = E_1 = 0$$

$$E_2 = ES_2 = ES_1 + t_{12} = 0 + 4 = 4$$

$$E_3 = ES_3 = ES_1 + t_{13} = 0 + 1 = 1$$

$$E_4 = ES_4 = \text{Max} (ES_2 + t_{24}, ES_3 + t_{34}, ES_2 + t_{24}) \\ = \text{Max} (1 + 1, 4 + 1) = 5$$

$$E_5 = (E_3 + t_{35}) = 1 + 6 = 7$$

$$E_6 = E_5 + t_{56} = 7 + 4 = 11$$

$$E_7 = E_5 + t_{57} = 7 + 8 = 15$$

$$E_8 = \text{Max} (E_6 + t_{68}, E_7 + t_{78}) \\ = \text{Max} (11 + 1, 15 + 2) = 17$$

$$E_9 = E_4 + t_{49} = 5 + 5 = 10$$

$$E_{10} = \text{Max} (E_9 + t_{9,10}, E_8 + t_{8,10}) \\ = \text{Max} (10 + 7, 17 + 5) = 22$$

Backward Pass Calculation: In this, we calculate the latest finish and the latest start time. The latest time  $L$  for an event  $i$  is given by  $L_i = \text{Min}_j (LF_j - t_{ij})$

Where  $LF_j$  is the latest finish time for the event  $j$ ,  $t_{ij}$  is the normal time of the activity.

$$L_{10} = 22$$

$$L_9 = L_{10} - t_{9,10} = 22 - 7 = 15$$

$$L_8 = L_{10} - t_{8,10} = 22 - 5 = 17$$

$$L_7 = L_8 - t_{7,8} = 17 - 2 = 15$$

$$L_6 = L_8 - t_{6,8} = 17 - 1 = 16$$

$$L_5 = \text{Min} (L_6 - t_{5,6}, L_7 - t_{5,7}) \\ = \text{Min} (16 - 4, 15 - 8) = 7$$

$$L_4 = L_9 - t_{4,9} = 15 - 5 = 10$$

$$L_3 = \text{Min} (L_4 - t_{3,4}, L_5 - t_{3,5})$$

$$= \text{Min} (10 - 1, 7 - 6) = 1$$

$$L_2 = L_4 - t_{2,4} = 10 - 1 = 9$$

$$L_1 = \text{Min} (L_2 - t_{1,2}, L_3 - t_{1,3}) = \text{Min} (9 - 4, 1 - 1) = 0.$$

These calculations are shown in the given table.

To find the TF (Total Float): Considering the activity 1-2, TF of (1-2) = Latest start-Earliest start.

$$\text{So, } TF = 5 - 0 = 5$$

$$\text{Similarly, } TF(2-4) = LS - ES$$

$$\text{So, } TF = 9 - 4 = 5$$

Free float = TF - Head event slack.

Consider the activity 1-2

$$FF \text{ of } 1-2 = TF \text{ of } 1-2 - \text{Slack for the head event } 2$$

$$\text{So, } FF = 5 - (9 - 4) \text{ (from the figure for event } 2)$$

$$\Rightarrow FF = 5 - 5 = 0$$

$$FF \text{ of } 2-4 = TF \text{ of } 2-4 - \text{Slack for the head event } 4$$

$$\text{So, } FF = 5 - (10 - 5) = 5 - 5 = 0$$

Like this we calculate the TF and FF for the remaining activities.

From the above table we observe that the activities 1-3, 3-5, 5-7, 7-8, 8-10 are the critical activities as their total float is 0.

Hence, we have the following critical path.

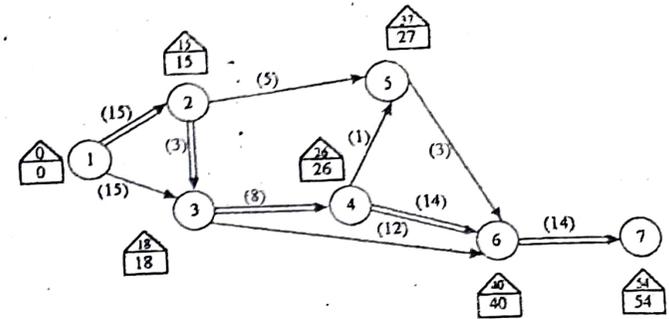
1 → 3 → 5 → 7 → 8 → 10, with the total project duration of 22 days.

**Example 5.9:** A small maintenance project consists of the following jobs whose precedence relationships is given below:

Job	1-2	1-3	2-3	2-5	3-4	3-6	4-5	4-6	5-6	6-7
Duration(days)	15	15	3	5	8	12	1	14	3	14

- Draw an arrow diagram representing the project.
- Find the total float for each activity.
- Find the critical path and the total project duration.

Solution:



**Forward Pass Calculation:** In this we estimate the earliest start and the earliest finish time  $ES_j$ , given by,

$ES_j = \text{Max} (ES_i + t_{ij})$  where  $ES_i$  is the earliest start time and  $t_{ij}$  is the normal time for the activity  $(i, j)$ .

$$ES_1 = 0$$

$$ES_2 = ES_1 + t_{12} = 0 + 15 = 15$$

$$ES_3 = \text{Max} (ES_2 + t_{23}, ES_1 + t_{13})$$

$$= \text{Max} (15 + 3, 0 + 15) = 18$$

$$ES_4 = ES_3 + t_{34} = 18 + 8 = 26$$

$$ES_5 = \text{Max} (ES_2 + t_{25}, ES_4 + t_{45})$$

$$= \text{Max} (15 + 5, 26 + 1) = 27$$

$$ES_6 = \text{Max} (ES_3 + t_{36}, ES_4 + t_{46}, ES_5 + t_{56})$$

$$= \text{Max} (18 + 12, 26 + 14, 27 + 3)$$

$$= 40$$

$$ES_7 = ES_6 + t_{67} = 40 + 14 = 54$$

**Backward pass calculation:** In this we calculate the latest finish and latest start time  $LF_j$ , given by  $LF_j = \text{Min} (LF_i - t_{ji})$  where  $LF_i$  is the latest finish time for the event  $j$ .

$$LF_7 = 54$$

$$LF_6 = LF_7 - t_{67} = 54 - 14 = 40$$

$$LF_5 = LF_6 - t_{56} = 40 - 3 = 37$$

$$LF_4 = \text{Min} (LF_5 - t_{45}, LF_6 - t_{46})$$

$$= \text{Min} (37 - 1, 40 - 14) = 26$$

$$LF_3 = \text{Min} (LF_4 - t_{34}, LF_6 - t_{36})$$

$$= \text{Min} (26 - 8, 40 - 12) = 18$$

$$LF_2 = \text{Min} (LF_3 - t_{23}, LF_5 - t_{25})$$

$$= \text{Min} (37 - 5, 18 - 3) = 15$$

$$LF_1 = \text{Min} (LF_2 - t_{12}, LF_3 - t_{13})$$

$$= \text{Min} (15 - 15, 18 - 15) = 0$$

The following table gives the calculation for critical path and total float.

Activity	Normal Time	Earliest		Latest		Total Float $LF_j - ES_i$ or $LF_i - ES_j$
		Start	Finish	Start	Finish	
		$ES_i$	$ES_j$	$LF_i$	$LF_j$	
1-2	15	0	15	0	15	0
1-3	15	0	15	3	18	3
2-3	3	15	18	15	18	0
2-5	5	15	20	32	37	17
3-4	8	18	26	18	26	0
3-6	12	18	30	28	40	10
4-5	1	26	27	36	37	10
4-6	14	26	40	26	40	0
5-6	3	27	30	37	40	10
6-7	14	40	54	40	54	0

From the above table we observe that the activities 1-2, 2-3, 3-4, 4-6, 6-7 are the critical activities and the critical path is given by, 1 → 2 → 3 → 4 → 6 → 7.

The total project completion is given by 54 days.

Example 5.10: Tasks A, B, ... H, I constitute a project. The notation X<Y means that the task X must be completed before Y is started. With the notation,

A<D; A<E; B<F; D<F; C<G; C<H; F<I; G<I

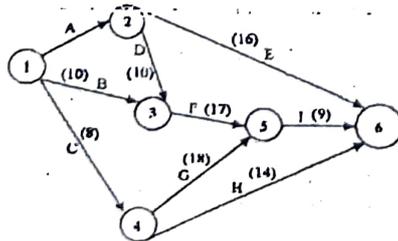
Draw a graph to represent the sequence of tasks and find the minimum time of completion of the project, when the time (in days) of completion of each task is as follows:

The above constraints can be given in the following table.

Task	A	B	C	D	E	F	G	H	I
Time (days)	8	10	8	10	16	17	18	14	9

Solution: The above constraints can be given in the following table.

Activity	A	B	C	D	E	F	G	H	I
Preceding Activity	-	-	-	A	A	B, D	C	B	F, G



**Time Calculation:** Using forward and backward pass calculation, we first estimate the earliest and the latest time for each event.

$$ES_1 = E_1 = 0$$

$$E_2 = E_1 + t_{12} = 0 + 8 = 8$$

$$E_3 = \text{Max}(E_1 + E_{13}, E_2 + t_{23})$$

$$= \text{Max}(0 + 10, 8 + 10) = 18$$

$$E_4 = E_1 + t_{14} = 0 + 8 = 8$$

$$E_5 = \text{Max}(E_3 + t_{35}, E_4 + t_{45})$$

$$= \text{Max}(18 + 17, 8 + 18) = 35$$

$$E_6 = \text{Max}(E_2 + t_{26}, E_4 + t_{46}, E_5 + t_{56})$$

$$= \text{Max}(8 + 16, 8 + 14, 35 + 9) = 44$$

The value of the latest time can now be obtained.

$$L_6 = E_6 = 44 \text{ (Target completion time for the project)}$$

$$L_5 = L_6 - t_{56} = 44 - 9 = 35$$

$$L_4 = \text{Min}(L_6 - t_{46}, L_5 - t_{45})$$

$$= \text{Min}(44 - 14, 35 - 18) = 17$$

$$L_3 = L_5 - t_{35} = 35 - 17 = 18$$

$$L_2 = \text{Min}(L_6 - t_{26}, L_3 - t_{23})$$

$$= \text{Min}(44 - 16, 18 - 10) = 8$$

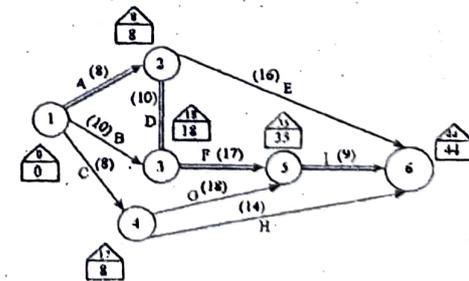
$$L_1 = \text{Min}(L_4 - t_{14}, L_3 - t_{13}, L_2 - t_{12})$$

$$= \text{Min}(17 - 8, 18 - 10, 8 - 8) = 0$$

To evaluate the critical events, all these calculations are put in the following table.

Task	Normal Time/Days	Earliest		Latest		Float		
		Start	Finish	Start	Finish	TF	FF	IF
A 1-2	8	0	8	0	8	0	0=0	0-0=0
B 1-3	10	0	10	8	18	8	8-0=8	8-0=8
C 1-4	8	0	8	9	17	9	9-0=9	0-0=0
D 2-3	10	8	18	8	18	0	0=0	0-0=0
E 2-6	16	8	24	28	44	20	20-0=20	20-0=20
F 3-5	17	18	35	18	35	0	0=0	0-0=0
G 4-5	18	8	26	17	35	9	9-0=9	9-0=0
H 4-6	14	8	22	30	44	22	22-0=22	22-9=13
I 5-6	9	35	44	35	44	0	0=0	0-0=0

The above table shows that the critical events are the tasks 1-2, 2-3, 3-5, 5-6 as their total float is zero.



The critical path is given by 1 → 2 → 3 → 5 → 6 or A → D → F → I with the total project duration as 44 days.

## 5.5 PROGRAMME EVALUATION AND REVIEW TECHNIQUE (PERT)

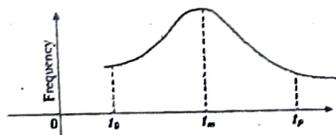
The network methods discussed so far may be termed as deterministic, since estimated activity times are assumed to be known with certainty. However, in research project or design of gear box of a new machine, various activities are based on judgement. It is difficult to obtain a reliable time estimate due to the changing technology. Time values are subject to chance variations. For such cases where the activities are non-deterministic in nature, PERT was developed. Hence, PERT is a probabilistic method where the activity times are represented by a probability distribution. This probability distribution of activity times is based upon three different time estimates made for each activity. These are as follows:

- (i) Optimistic time estimate
- (ii) Most likely time estimate
- (iii) Pessimistic time estimate

**Optimistic Time Estimate:** It is the smallest time taken to complete the activity if everything goes on well. There is very little chance that activity can be done in time less than the optimistic time. It is denoted by  $t_o$  or  $a$ .

**Most Likely Time Estimate:** It refers to the estimate of the normal time the activity would take. This assumes normal delays. It is the mode of the probability distribution. It is denoted by  $t_m$  or  $(m)$ .

**Pessimistic Time Estimate:** It is the longest time that an activity would take if everything goes wrong. It is denoted by  $t_p$  or  $b$ . These three time values are shown in the following figure.



Time Distribution Curve

From these three time estimates, we have to calculate the expected time of an activity. It is given by the weighted average of the three time estimates,

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

$\beta$  distribution with weights of 1, 4, 1, for  $t_o$ ,  $t_m$  and  $t_p$  estimates respectively.

Variance of the activity is given by,

$$\sigma^2 = \left[ \frac{t_p - t_o}{6} \right]^2$$

The expected length (duration), denoted by  $T_e$  of the entire project is the length of the critical path, i.e., the sum of the  $t_e$ 's of all the activities along the critical path.

The main objective in the analysis through PERT is to find the completion for a particular event within specified date  $T_p$  given by  $P(Z \leq D)$  where,

$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

Where  $Z$  stands for standard normal variable.

### 5.5.1 PERT Procedure

**Step 1:** Draw the project network.

**Step 2:** Compute the expected duration of each activity using the formula.

$$t_e = \frac{t_o + 4t_m + t_p}{6}$$

Also calculate the expected variance  $\sigma^2$  of each activity.

$$\text{i.e., } \sigma^2 = \left( \frac{t_p - t_o}{6} \right)^2$$

**Step 3:** Compute the earliest start, earliest finish, latest start, latest finish and total float of each activity.

**Step 4:** Find the critical path and identify the critical activities.

**Step 5:** Compute the project length variance  $\sigma^2$  which is the sum of the variance of all the critical activities and hence find the standard deviation of the project length  $\sigma$ .

**Step 6:** Calculate the standard normal variable  $Z = \frac{T_s - T_e}{\sigma}$  where  $T_s$  is the scheduled time to complete the project.

$T_e$  = Normal expected project length duration.

$\sigma$  = Expected standard deviation of the project length.

Using the normal curve, we can estimate the probability of completing the project within a specified time.

**Example 5.11:** The following table shows the jobs of a network alongwith their time estimates.

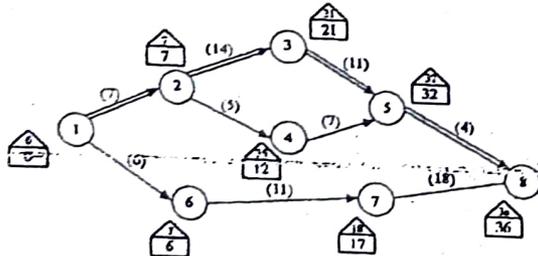
Job	1-2	1-6	2-3	2-4	3-5	4-5	6-7	5-8	7-8
$a$ (days)	1	2	2	2	7	5	5	3	8
$m$ (days)	7	5	14	5	10	5	8	3	17
$b$ (days)	13	14	26	8	19	17	29	9	32

Here,  $a$  is optimistic time,  $m$  is most likely time and  $b$  is pessimistic time estimate.

Draw the project network and find the probability that the project is completed in 40 days.

Solution: First we calculate the expected time and standard deviation for each activity.

Activity	$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1-2	$\frac{1+4 \times 7+13}{6} = 7$	$\left(\frac{13-1}{6}\right)^2 = 4$
1-3	$\frac{2+4 \times 5+14}{6} = 6$	$\left(\frac{14-2}{6}\right)^2 = 4$
2-3	$\frac{2+4 \times 14+26}{6} = 14$	$\left(\frac{26-2}{6}\right)^2 = 16$
2-4	$\frac{2+5 \times 4+8}{6} = 5$	$\left(\frac{8-2}{6}\right)^2 = 1$
3-5	$\frac{7+4 \times 10+19}{6} = 11$	$\left(\frac{19-7}{6}\right)^2 = 4$
4-5	$\frac{5+5 \times 4+17}{6} = 7$	$\left(\frac{17-5}{6}\right)^2 = 4$
6-7	$\frac{5+8 \times 4+29}{6} = 11$	$\left(\frac{29-5}{6}\right)^2 = 16$
5-8	$\frac{3+3 \times 4+9}{6} = 4$	$\left(\frac{9+3}{6}\right)^2 = 1$
7-8	$\frac{8+4 \times 17+32}{6} = 18$	$\left(\frac{32-8}{6}\right)^2 = 16$



Expected project duration = 36 days.  
 Critical path 1 → 2 → 3 → 5 → 8  
 Project length variance =  $\sigma^2 = 4 + 16 + 4 + 1 = 25$   
 $\sigma = 5$

Probability that the project will be completed in 40 days is given by,

$$P(Z \leq D)$$

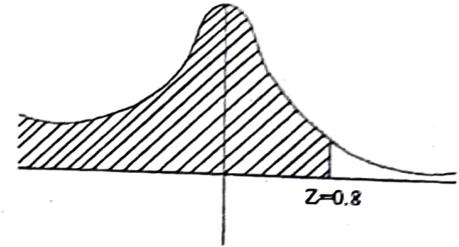
$$D = \frac{T_s - T_e}{\sigma} = \frac{40 - 36}{5} = \frac{4}{5} = 0.8$$

Area under the normal curve for  $\Delta = 0.8$ ,

$$P(Z \leq 0.8)$$

$$= 0.5 + \phi(0.8) \quad [\phi(0.8) = 0.2881 \text{ (refer Z-table)}]$$

$$= 0.5 + 0.2881 = 0.7881 = 78.81\%$$



Conclusion: If the project is performed 100 times under the same conditions, there will be 78.81 occasions for this job to be completed in 40 days.

Example 5.12: A small project is composed of seven activities whose time estimates are listed in the table as follow.

Activity	Estimated Duration (Weeks)		
	Optimistic	Most Likely	Pessimistic
1-2	1		7
1-3	1	4	7
2-4	2	2	8
2-5	1	1	1
3-5	2	5	14
4-6	2	5	8
5-6	3	6	15

You are required to:

- Draw the project network.
- Find the expected duration and variance of each activity.
- Calculate the early and late occurrence for each event and the expected project length.
- Calculate the variance and standard deviations of project length.
- What is the probability that the project will be completed:
  - 4 weeks earlier than expected.
  - Not more than 4 weeks later than expected.
  - If the project due date is 19 weeks, what is the probability of meeting the due date.

Solution: The expected time and variance of each activity is computed as shown in the table below:

NOTES

Activity	a	m	b	$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
1-2	1	1	7	2	1
1-3	1	4	7	4	1
1-4	2	2	8	3	1
2-5	1	1	1	1	0
3-5	2	5	14	6	4
4-6	2	5	8	5	1
5-6	3	6	15	7	4

The earliest and the latest occurrence time for each is calculated as below:

$$E_1 = 0; E_2 = 0 + 2 = 2$$

$$E_3 = 0 + 4 = 4$$

$$E_4 = 0 + 3 = 3$$

$$E_5 = \text{Max}(2 + 1, 4 + 6) = 10$$

$$E_6 = \text{Max}(10 + 7, 3 + 5) = 17$$

To determine the latest expected time we start from  $E_6$  being the last event and move backwards subtracting  $t_e$  from each activity. Hence, we have

$$L_6 = E_6 = 17$$

$$L_5 = L_6 - 7 = 17 - 7 = 10$$

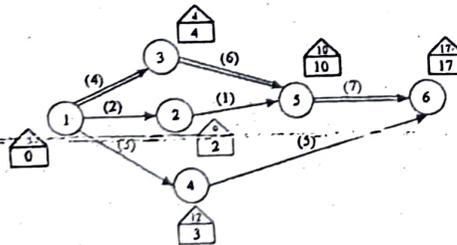
$$L_4 = 17 - 5 = 12$$

$$L_3 = 10 - 6 = 4$$

$$L_2 = 10 - 1 = 9$$

$$L_1 = \text{Min}(9 - 2, 4 - 4, 12 - 3) = 0$$

Using the above information, we get the following network, where the critical path is shown by the double line arrow.



We observe the critical path of the above network as 1 → 3 → 5 → 6.

The expected project duration is 17 weeks, i.e.,  $T_e = 17$  weeks.

The variance of the project length is given by,

$$\sigma^2 = 1 + 4 + 4 = 9$$

Hence,

$$\sigma = 3$$

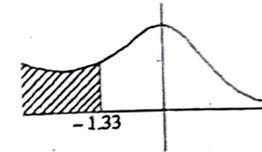
(i) The probability of completing the project within 4 weeks earlier than expected is given by,

$$P(Z \leq D) \text{ where } D = \frac{T_s - T_e}{\sigma}$$

$$D = \frac{\text{Due date} - \text{Expected date of completion}}{\sqrt{\text{Project variance}}}$$

$$D = \frac{17 - 4 - 17}{3} = \frac{13 - 17}{3} = \frac{-4}{3} = -1.33$$

$$\begin{aligned} \therefore P(Z \leq -1.33) &= 0.5 - \phi(1.33) \\ &= 0.5 - 0.4082 \text{ (from the table)} \\ &= 0.0918 = 9.18\% \end{aligned}$$



Conclusion: If the project is performed 100 times under the same conditions, then there will be 9 occasions for this job to be completed in 4 weeks earlier than expected.

(ii) The probability of completing the project not more than 4 weeks later than expected is given by,

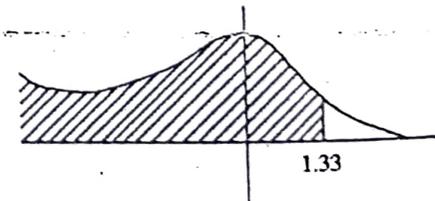
$$P(Z \leq D)$$

Where,  $D = \frac{T_s - T_e}{\sigma}$

Here,  $T_s = 17 + 4 = 21$

$$D = \frac{21 - 17}{3} = \frac{4}{3} = 1.33$$

$$\begin{aligned} P(Z \leq 1.33) &= 0.5 + \phi(1.33) \\ &= 0.5 + 0.4082 \text{ (from the table)} \\ &= 0.9082 = 90.82\% \end{aligned}$$



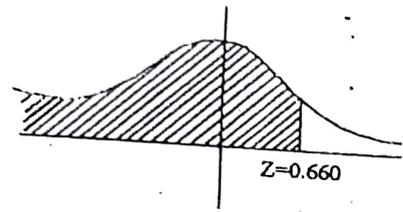
Conclusion: If the project is performed 100 times under the same conditions, then there will be 90.82 occasions when this job will be completed not more than 4 weeks later than expected.

(iii) The probability of completing the project within 19 weeks, is given by,

$$\begin{aligned} P(Z \leq D) \text{ where, } D &= \frac{19 - 17}{3} = \frac{2}{3} \\ &= 0.666 \end{aligned}$$

[Since  $T_s = 19$ ]

$= 0.5 + 0.2514$  (from the table)  
 $= 0.7514 = 75.14\%$



**Conclusion:** If the project is performed 100 times under the same conditions, then there will be 75.14 occasions for this job to be completed in 19 weeks.

**Example 5.13:** Consider the following project.

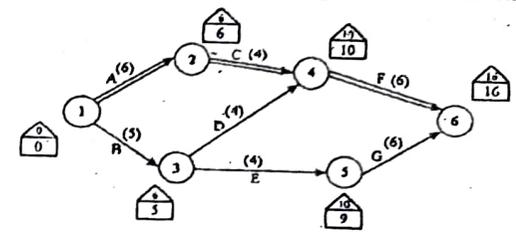
Activity	Time Estimate in Weeks			Predecessor
	$t_o$	$t_m$	$t_p$	
A	3	6	9	None
B	2	5	8	None
C	2	4	6	A
D	2	3	10	B
E	1	3	11	B
F	4	6	8	C, D
G	1	5	15	E

Find the path and standard deviation. Also find the probability of completing the project by 18 weeks.

**Solution:** First we calculate the expected time and variance of each activity as in the following table:

Activity	$t_o$	$t_m$	$t_p$	$t_e = \frac{t_o + 4t_m + t_p}{6}$	$\sigma^2 = \left(\frac{t_p - t_o}{6}\right)^2$
A	3	6	9	$\frac{3 + 4 \times 6 + 9}{6} = 6$	$[(9 - 3)/6]^2 = 1$
B	2	5	8	$\frac{30}{6} = 5$	$[(8 - 2)/6]^2 = 1$
C	2	4	6	$24/6 = 4$	$[(6 - 2)/6]^2 = 0.444$
D	2	3	10	4	1.777
E	1	3	11	4	2.777
F	4	6	8	6	0.444
G	1	5	15	6	5.444

We construct the network with the help of predecessor relation given in the data.



Critical path is  $1 \rightarrow 2 \rightarrow 4 \rightarrow 6$  or  $A \rightarrow C \rightarrow F$

The project length = 16 weeks.

Project length variance  $\sigma^2 = 1 + 0.444 + 0.444 = 1.888$

Standard deviation =  $\sigma = 1.374$

The probability of completing the project in 18 weeks is given by:

$P(Z \leq D)$

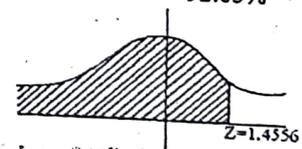
Where,

$D = \frac{T_s - T_e}{\sigma}$

$T_s = 18; T_e = 16; \sigma = 1.374$

$D = \frac{18 - 16}{1.374} = 1.4556$

$P(Z \leq D) = P(Z \leq 1.4556) = 0.5 + \phi(1.4556)$   
 $= 0.5 + 0.4265$  (from table)  
 $= 0.9265 = 92.65\%$



**Conclusion:** If the project is performed 100 times under the same conditions, then there will be 92.65 occasions when this job will be completed by 18 weeks.

**Example 5.14:** Assuming that the expected times are normally distributed, find the probability of meeting the schedule date as given for the network.

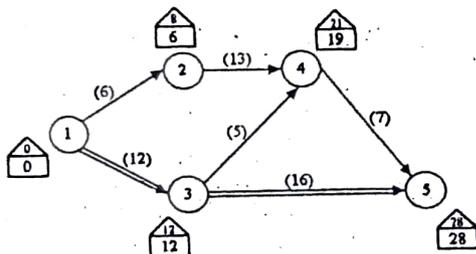
Activity (i-j)	Days		
	Optimistic $t_o$	Most Likely $t_m$	Pessimistic $t_p$
1-2	2	5	14
1-3	9	12	15
2-4	5	14	17
3-4	4	4	10
4-5	8	17	20
3-5	6	6	12

Scheduled project completion date is 30 days. Also, find the date on which the project manager can complete the project with a probability of 0.90.

**Solution:** The expected time  $t_e$  and variance for each activity is calculated in the following table:

Activity	$t_e = (t_o + 4t_n + t_p)/6$	$\sigma^2 = ((t_p - t_o)/6)^2$
1-2	6	4
1-3	12	1
2-4	13	4
3-4	5	1
3-5	16	4
4-5	7	1

To determine the critical path, the earliest expected time and the latest allowable time. First we draw the project network as follows:



The critical path is given by 1 → 3 → 5 and the project duration is given by 28 days. Project length variance =  $\sigma^2 = 1 + 4 = 5$ . Standard deviation =  $\sqrt{\sigma^2} = 2.236$ .

The probability of completing the project within 30 days is given by,

$$P(Z \leq D), \text{ where } D = \frac{T_s - T_e}{\sigma} = \frac{30 - 28}{2.236} = 0.8944$$

$$\begin{aligned} P(Z \leq 0.8944) &= 0.5 + \phi(0.8944) \\ &= 0.8133 \\ &= 81.33\% \end{aligned}$$

**Conclusion:** If the project is performed 100 times under the same conditions, then there will be 81.33 occasions when the project will be completed in 30 days.

If the probability for the completion of the project is 0.90 then the corresponding value of  $Z = 1.29$ .

$$Z = \frac{T_s - T_e}{\sigma} = 1.29$$

$$\text{i.e., } \frac{T_s - 28}{2.236} = 1.29$$

$$\therefore T_s = (1.29)(2.236) + 28$$

$$\therefore T_s = 30.88 \text{ weeks}$$

## 5.6 COMPARISON AND LIMITATIONS OF PERT AND CPM

1. CPM is activity oriented, i.e., CPM network is built on the basis of activities. Also results of various calculations are considered in terms of activities of the project. On the other hand, PERT is event oriented.
2. CPM is a deterministic model, i.e., it does not take into account the uncertainties involved in the estimation of time for execution of a job or an activity. It completely ignores the probabilistic element of the problem. PERT however is a probabilistic model. It uses three estimates of the activity time- optimistic, pessimistic and most likely with a view to take into account time uncertainty. Thus the expected duration of each activity is probabilistic indicates that there is fifty percent probability of getting the job done within the time.
3. CPM places dual emphasis on time and cost and evaluates the trade off between project cost and project time. It allows the project manager to manipulate project duration within certain limits so that project duration can be shortened to an optimal cost. On the other hand PERT is primarily concerned with time. It helps the manager to schedule and coordinate various activities so that the project can be completed on scheduled time.
4. Since the Critical Path Method does not account for uncertainty, it is best used in projects where the activity time estimate can be predicted fairly accurately. For example, for repetitive projects you can estimate the time for each activity quite accurately from past experience. Whereas for projects that have a higher degree of uncertainty, use the PERT Network. Most software projects will require you to account for a high degree of uncertainty.
5. Another difference in PERT and CPM is in how the diagrams are drawn. In PERT, events are placed in circles or rectangles to emphasize a point in time. Tasks are indicated by the lines connecting the network of events. In CPM the emphasis is on the tasks, which are placed in circles. The circles are then connected with lines to indicate the relationship between the tasks. CPM use has become more widespread than the use of PERT applications.

PERT and CPM are used together because they have similarities. For example, PERT and CPM both assume that a small set of activities, which make up the longest path through the activity network control the entire project. In addition to that, PERT and CPM also share the following six key assumptions:

1. All tasks have distinct beginning and end points.
2. All estimates can be mathematically derived.
3. Tasks must be able to be arranged in a defined sequence that produces a pre defined result.
4. Resources may be shifted to meet the need.
5. Cost and time share a direct relationship, i.e., cost of each activity is evenly spread over time.
6. Time, of itself, has no value.

- A range of time estimates (by PERT).
- Likely time estimates (by PERT and CPM).
- Cost estimates (by CPM).
- Time and costs if crashed (by CPM).
- Probabilities of completion on time for a range of times (by PERT).
- A clear path of tasks that are critical to the project (by PERT and CPM).
- A central focus for solid communications on project issues (by PERT and CPM).

### Limitations of PERT/CPM

- Clearly defined, independent and stable activities.
- Specified precedence relationships.
- Over emphasis on critical paths.
- Deterministic CPM model.
- Activity time estimates are subjective and depend on judgment. If the estimates are subjective, then it compromises the purpose of the formula. The weighted estimate and standard deviation will not accurately depict the amount of time required for each task. In case where there is little experience in performing the activity, these estimates may be only a guess. Moreover if the person or group performing the activity estimates the time, there may be a bias in the estimate.
- PERT assumes a beta distribution for these time estimates, but the actual distribution may be different.
- Even if the beta distribution assumption holds, PERT assumes that the probability distribution of the project completion time is the same as that of the critical path. PERT consistently underestimates the expected project completion time due to alternate paths becoming critical. Under estimation of time can cause huge problems in project management. Not only can it cause the project to fall behind, but it can also cause overages in budget when employees are either forced to pull overtime to meet project deadlines or the project might over extend what was budgeted resource wise, thus causing a problem with over allocation.

## 5.7 BASIC CONCEPTS OF CRASHING

Crashing of a project means intentionally reducing the duration of a project by allocating more resources to it. A project can be crashed by crashing its critical activities (because the duration of a project is dependent upon the duration of its critical activities). The use of more workers, better equipment, overtime, etc., would generate higher direct costs for individual activities. However, shortening the overall time of the project would also reduce certain fixed and overhead expenses of supervision, as well as indirect costs that vary with the length of the project.

We know that by adding more resources, the duration of an activity can be reduced. If an activity gets completed in ten days with five men working on it, the same activity can be finished in say, six days with ten men (exact mathematical relationships do not work here) working on it. The initial direct-cost was 50 man-days (5 men x 10 days) and now it is 60 man-days (10 men x 6 days). Therefore, the direct cost has increased by 10 man-days.

At the same time, because of the decrease in duration of the activity by four days, the indirect cost (cost of supervision) decreases. Hence, we can conclude that the direct and indirect costs are inversely proportional to each other, i.e., when one increases, the other decreases.

An activity can be crashed by adding more resources only up to a definite limit. Beyond this limit, the duration of the activity does not decrease by adding more resources. This is due to decreasing efficiency of labour, and also increasing confusion due to a large number of resources. In our example, if we increase the number of workers to 15, the same activity can probably be done in four days; but by adding ten more men (so that 20 men work on this activity), the activity time may not decrease further.

The crash time is the shortest time that could be achieved if all efforts (at any reasonable cost) were made to reduce the activity time. The limit beyond which the duration of the activity does not decrease by adding any amount of resources is called the crash time. It is the shortest possible activity time.

The direct cost associated with each crash time is called the crash cost.

The normal time (10 days in our example) can be defined as the duration of an activity when the minimum possible resources required for its performance are deployed. The lowest expected activity costs are called the normal costs.

Project direct cost is the direct cost involved in all the activities of the project.

Project indirect cost is the costs associated with sustaining a project. It includes the cost of supervision during the implementation of the project, overheads, facilities, penalty costs and lost incentive payments. The salaries paid to the project manager/supervisor, etc., miscellaneous costs due to delays in the project, and rewards to the project team members for its early completion are indirect costs. Project indirect cost is dependent upon other length of duration of the project. A project having a longer duration will have a higher indirect cost (due to supervision required for longer duration).

In any project, there is a direct relationship between the time taken to complete an activity and the project cost. On one hand, it costs money to expedite a project. Costs associated with expediting a project are called activity direct costs, and are different from project direct costs. Some examples of activity direct costs are—hiring more workers, buying or leasing additional equipment, drawing on additional support facilities, etc.

If activity direct costs rise, project indirect costs will fall. Therefore, in a real situation, we need to have a time-cost trade-off; this means the sum of activity direct costs and project indirect costs must be minimum.

During the process of crashing a project, the critical path may get changed. At some stage of crashing, there may even be two or more critical paths (having the

### Check Your Progress

What is a critical activity and critical path?

What is PERT? Where is it used?

What are optimistic time, most likely time and pessimistic time for activities in a project?

What is expected time for an activity in a project?

What is variance of an activity?

What is the standard normal variable of a project?

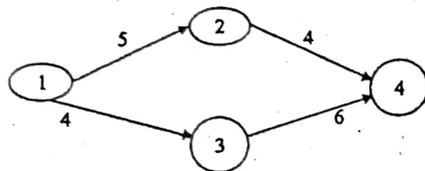
Why are PERT and CPM used together?

same duration) simultaneously. In such situations, one activity is chosen from each of the critical paths and these activities are crashed by unit time to reduce the duration of the project by unit time.

Time-cost models search for the optimum reductions in time. We seek to shorten the length of a project to the point where the savings in indirect project costs is offset by the increased direct expenses incurred in the individual activities.

**Example 5.15:** A network has four activities with expected times as shown. The minimum feasible times and cost per day to gain reductions in the activity times are also shown. If fixed project costs are ₹ 90 per day, what is the lowest cost time schedule?

(All costs are in ₹ '000)



Activity	Minimum Time in Days	Direct Costs of Time Reduction (₹)
1-2	2	40 (each day)
1-3	2	35 (first day, 80 (second day)
2-4	4	None possible
3-4	3	45 (first day), 110 (other days)

**Solution:**

First we must determine the critical path and critical path time cost.

Path	Path Times	Total Project Cost
1-2, 2-4	5 + 4 = 9	
1-3, 3-4	4 + 6 = 10	10 days × ₹ 90/day = ₹ 900

For ease of reference, let us call the paths A and B respectively. Path B that is, 1-3-4 is the critical path with duration 10 days and cost ₹ 900.

Next, we must select the activity that can reduce critical path time at the least cost. Select activity 1-3 at ₹ 35 per day, which is less than the ₹ 90 per day fixed cost. Reduce activity 1-3 to 3 days. Revise the critical path time cost.

Revised Path Times	Total Fixed Cost	Savings over Previous Schedule
A: 5 + 4 = 9	9 × ₹ 90 = ₹ 810	₹ 900 - (810 + 35) = ₹ 55
B: 3 + 6 = 9		

Both paths are now critical, so we must select an activity on each path. Select activity 1-2 at ₹ 40 per day and 3-4 at ₹ 45 per day. Reduce activity 1-2 to 4 days and 3-4 to 5 days. Revise the critical path time cost.

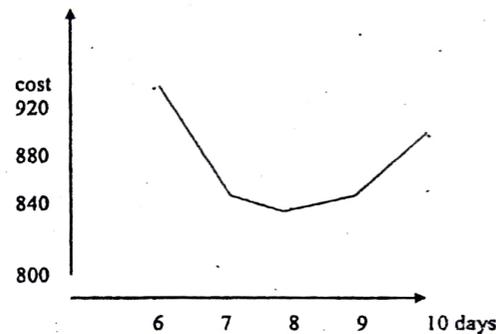
Revised Path Times	Total Fixed Cost	Savings over Previous Schedule
A: 4 + 4 = 8	8 × 90 = ₹ 720	810 - (720 + 40 + 45) = ₹ 5
B: 3 + 5 = 8		

Let us see if we can reduce the time of both paths any further. Activity 1-2 is a good candidate on Path A, for it is still at 4 days and can go to 3 for a ₹ 40 cost. But when this cost is combined with the ₹ 80 cost for reducing activity 1-3 another day, the sum is greater than ₹ 90, so further reduction is not economically justified.

The final step in time-cost analysis is to compare the crash times and the costs associated with them (crash costs). A sufficient number of intermediate schedules are computed such that the total of the direct and indirect (fixed) project costs can be plotted.

**Example 5.16:** Graph the total relevant costs for the previous example, and indicate the optimal time-cost trade-off value.

Project Length (days)	Indirect Cost	Activity Reduced	Relevant Direct Cost	Relevant Total Cost
10	900	None	0	900
9	810	1-3	35	810 + 35 = 845
8	720	1-2 and 3-4	35 + 85 = 120	720 + 40 + 45 + 35 = 840
7	630	1-2 and 1-3	120 + 120 = 240	870
6	540	1-2 and 3-4	240 + 150 = 390	930



This graph is called the crash-time diagram for completing the project. The lowest total cost is to complete the project in 8 days at a cost of ₹ 840,000. However, extending it to 9 days adds only ₹ 5000 to this cost.

## 5.8 RESOURCE LEVELING AND RESOURCE SMOOTHING

Resource allocation (also known as resource scheduling) implies the task of allocation of resources to various activities in such a manner that the allocation is considered as acceptable under the given situation. The task of allocation of resources is important as the final schedule depends upon the quantity of deployment of resources. The basic question then is, How should the resources be allocated?

This depends upon several factors such as availability of resources, requirements restrictions in regard to completion date, and so on. Various types of problems may be encountered in this connection, but we shall consider only two of such problems:

(i) Resource levelling (ii) Limited resource allocation.

(i) **Resource levelling** (also known as local smoothing) means the resource scheduling exercise in which the resource demand is evened out or levelled as much as possible. In other words, resource levelling refers to the scheduling of activities within the limits of the available floats in such a way that variations in resources requirements are minimized.

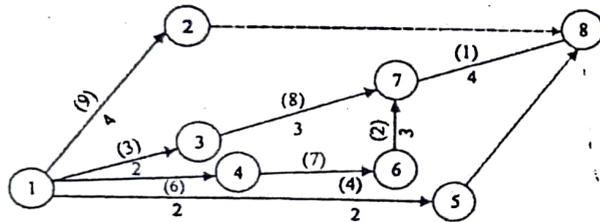
(ii) **Limited resource allocation** We often find that projects require costly items of plant and equipment for the execution of work of which only a limited number

is available. Such limited resources must be allocated with a lot of care so that the total requirement should not exceed the ceiling and the utilization factor remains high. This necessitates rescheduling of some or all of the activities and may even involve delay in the overall completion of the project.

For resource levelling the methodology involves the following steps:

- (i) Prepare the list of resources that would be required for the execution of various activities.
- (ii) Prepare the resource profiles for each resource by resource aggregation exercise.
- (iii) Identify the periods of peak and low demands.
- (iv) Make an attempt to lower the peak to fill up the troughs, i.e., to make the demand as uniform as possible. This can be done by altering the times of start and finish of non-critical activities in accordance of their floats without affecting the overall completion date of the project:

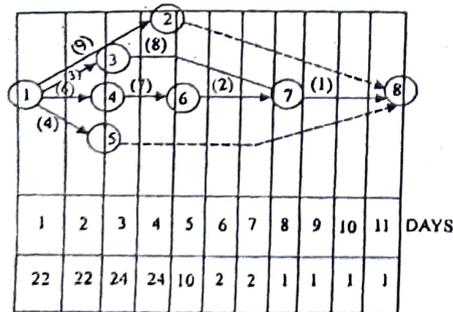
**Example 5.17:** Work out the aggregate resource requirements, day by day for the following network.



The figures over the arrows in brackets indicate the requirement of labour and figures below the arrows show the duration of an activity (in days). Presuming no resource constraint, smooth out the requirements of the resource (i.e., labour) to the extent possible and outline the revised schedule.

**Solution:**

For finding the aggregate resource requirements day by day, it would be helpful to present the given network as shown below:

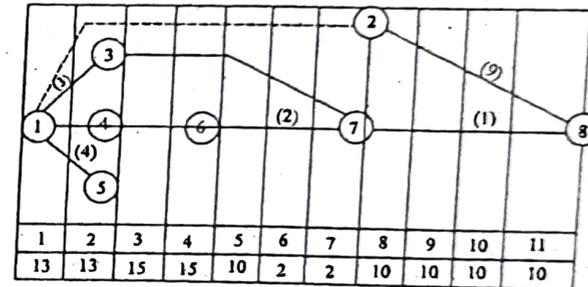


The manpower requirements vary from 1-24 and variations of this size are not acceptable. So it is necessary to smooth out the aggregate requirements. This can be done by modifying the input of resources to non-critical activities. To begin with

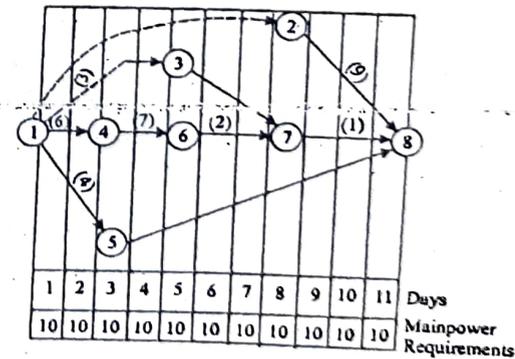
activities 1-4, 4-6, 6-7, and 7-8 are critical and have to be completed without delay. Activities 1-2, 1-3, 3-7 and 1-5 are non-critical activities. They can be delayed with their floats. Accordingly, let us now compute the float of each activity.

Activity	Start		Finish		Float
	ET	LT	ET	LT	
1-2	0	7	4	11	7
1-3	0	2	2	4	2
1-4	0	0	2	2	0
1-5	0	2	2	4	2
3-7	2	4	4	7	2
4-6	2	2	5	4	0
6-7	4	4	7	7	0
7-8	7	7	11	11	0

Activity 1-2 can be delayed for 7 days without affecting the overall project completion time. If we postpone this activity for 7 days we get the following schedule:



Thus, with above stated modification, the variation in manpower requirements has been smoothed from 1-24 to 2-15. We can make some further modifications as well. Activity 1-3 can be delayed for 2 days without affecting the overall project completion time and if we do so, we get the following schedule:



Thus, the pattern of resource demand has further improved. Perfect levelling has been made possible in this particular case.

The methodology for limited resource problem involves the following:

- (i) Attempt to lower the peak requirements of the resources by staggering the inputs on non-critical activities. At times, it may become necessary to tackle

## NOTES

- some critical activities for the purpose because of limited resource constraint and this may mean even delays in completion of the work.
- (ii) It is advantageous to re-arrange the activities in the descending order of the magnitude of their floats as resources can be diverted from the activities having large amounts of floats. Critical activities should only be tackled in the end, if necessary.
  - (iii) The following four rules (heuristics) for resource allocation in project scheduling with limited resources facilitate the work:
    - (a) Allocate the resources to activities serially in line (i.e. in the order the activities become due for start.)
    - (b) When several activities compete for the same resources give preference to activities having minimum float.
    - (c) If two competing activities happen to have equal floats, allocate the resources to the activity having lower duration.
    - (d) Reschedule non-critical jobs; it possible to free resources for scheduling critical jobs.

## 5.9 SUMMARY

- Network analysis is a method of planning and controlling projects by recording their interdependence in a diagrammatic form that enables each fundamental problem involved to be tackled separately.
- Network analysis clearly shows the interdependences between jobs to be performed in context of a project and thus enables people to see not only the overall plan but the ways in which their own activities depend upon or influence those of others.
- By splitting up the project into smaller activities, network analysis assists in the estimation of their durations leading to more accurate target dates. Network analysis allows schedules to be based on considerations of costs so as to complete projects in a given time at minimum expense.
- Network analysis separates the planning of the sequence of jobs from the scheduling of times for the jobs and thus it makes planning and scheduling effective.
- Network analysis is specially suited for projects which are not routine or repetitive and which will be conducted only once or a few times. Two most popular forms of this technique now used in many scheduling situations are the Critical Path Method (or simply CPM) and the Programme Evaluation and Review Technique (popularly known as PERT).
- Critical Path Method or CPM was developed to aid in the scheduling of routine plant overhaul, maintenance and construction work. This method differentiates between planning and scheduling.
- Planning refers to the determination of activities that must be accomplished and the order in which such activities should be performed to achieve the

objective of the project whereas scheduling refers to the introduction of time into the plan thereby creating a time table for the various activities to be performed.

- Programme Evaluation and Review Technique or PERT was developed for use in defence projects specifically but now this technique assists a business manager in planning and controlling a project.
- In PERT, the time assumed to perform each activity is uncertain and as such three time estimates (the optimistic, the pessimistic and the most likely) are used. It is often described as an approach of multiple time estimates to scheduling problems of long-range research and development projects.
- A project is defined as a combination of interrelated activities, all of which must be executed in a certain order for its completion.
- Network is the graphic representation of logically and sequentially connected arrows and nodes representing activities and events of a project. Networks are also called arrow diagram.
- An activity represents some action and is a time consuming effort necessary to complete a particular part of the overall project. Thus, each and every activity has a point of time where it begins and a point where it ends.
- The beginning and end points of an activity are called events or nodes. Event is a point in the time and does not consume any resource. It is represented by a numbered circle.
- Activities, which must be accomplished before a given event can occur are termed as preceding activities. Activities, which cannot be accomplished until an event has occurred are termed as succeeding activities. Activities, which can be accomplished concurrently are known as concurrent activities.
- Certain activities which neither consume time nor resources but are used simply to represent a connection or a link between the events are known as dummies. It is shown in the network by a dotted line.
- In a network diagram looping error is also known as cycling error. Drawing an endless loop in a network is known as error of looping.
- To disconnect an activity before the completion of all the activities in a network diagram is known as dangling.
- As per Fulkerson's rule, after the network is drawn in a logical sequence every event is assigned a number. The number sequence must reflect the flow of the network.
- Once the network of a project is constructed, the time analysis of the network becomes essential for planning various activities of the project. An activity time is a forecast of the time an activity is expected to take from its starting point to its completion (under normal conditions).
- An activity is said to be critical if a delay in its start will cause a further delay in the completion of the entire project.
- The sequence of critical activities in a network is called the critical path. It is the longest path in the network from the starting event to the ending event and

Check Your Progress

Define the term crashing.

What is crash time?

Define the term resource allocation.

defines the minimum time required to complete the project. In the network, it is denoted by double line.

- The critical path highlights those activities which must be performed more rapidly if the total project completion time is to be reduced.
- The term slack is normally associated with events. It indicates the amount of latitude that is available for an event to occur.
- Crashing of a project means intentionally reducing the duration of a project by allocating more resources to it.
- A project can be crashed by crashing its critical activities (because the duration of a project is dependent upon the duration of its critical activities). The use of more workers, better equipment, overtime, etc, would generate higher direct costs for individual activities.
- An activity can be crashed by adding more resources only up to a definite limit. Beyond this limit, the duration of the activity does not decrease by adding more resources. This is due to decreasing efficiency of labour, and also increasing confusion due to a large number of resources.
- The crash time is the shortest time that could be achieved if all efforts (at any reasonable cost) were made to reduce the activity time. The limit beyond which the duration of the activity does not decrease by adding any amount of resources is called the crash time. It is the shortest possible activity time.
- The direct cost associated with each crash time is called the crash cost.
- Project direct cost is the direct cost involved in all the activities of the project. Project indirect cost is the costs associated with sustaining a project. It includes the cost of supervision during the implementation of the project, overheads, facilities, penalty costs and lost incentive payments.
- During the process of crashing a project, the critical path may get changed. At some stage of crashing, there may even be two or more critical paths (having the same duration) simultaneously. In such situations, one activity is chosen from each of the critical paths and these activities are crashed by unit time to reduce the duration of the project by unit time.
- Resource allocation (also known as resource scheduling) implies the task of allocation of resources to various activities in such a manner that the allocation is considered as acceptable under the given situation.
- Resource levelling (also known as local smoothing) means the resource scheduling exercise in which the resource demand is evened out or levelled as much as possible. In other words, resource levelling refers to the scheduling of activities within the limits of the available floats in such a way that variations in resources requirements are minimized.
- Limited resource allocation is used to find that projects require costly items of plant and equipment for the execution of work of which only a limited number is available. Such limited resources must be allocated with a lot of care so that the total requirement should not exceed the ceiling and the utilization factor remains high. This necessitates rescheduling of some or all of the activities and may even involve delay in the overall completion of the project.

## 5.10 KEY TERMS

- **Network:** A graphic representation of logically connected activities and events where activities are presented as arrows and events as nodes
- **Activity:** An activity represents an action and an effort that consumes time that is needed to complete a part of the overall project
- **Preceding activity:** An activity that must be accomplished for an event to occur
- **Succeeding activity:** An activity that cannot occur until an event has occurred
- **Concurrent activities:** Activities which can be accomplished concurrently
- **Critical activity:** An activity is critical if delay in its start will cause further delay in completion of the entire project
- **Event:** An event is either start or end of an activity
- **Critical path:** It is the path connecting all critical events of the project from start to the completion of the project
- **PERT:** It stands for Program Evaluation Review Technique. It is a probabilistic method where activity times are represented by a probability distribution
- **CPM:** It stands for Critical Path Method and is based on determination of the critical path
- **Dummy activity:** An activity that consumes neither any resource nor time, but it is there on the network to show a link between events
- **Redundancy:** A dummy activity that is the only activity emanating from an event is a redundant activity which can be eliminated
- **Optimistic time:** The minimum time taken to complete an activity assuming that everything goes well
- **Most likely time:** The normal time taken by an activity assuming normal delay
- **Pessimistic time:** The longest time required for an activity to complete assuming everything going wrong
- **Expected time:** The time for an activity in a project, denoted in a PERT chart and is given by the formula, as given below:

$$\text{Expected time} = (\text{Optimistic time} + 4 \times \text{Most likely time} + \text{Pessimistic time}) / 6.$$

## 5.11 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Management functions involved in three phases of work involved in a project are, planning, scheduling and controlling.
2. Planning is setting of objectives of the project by listing of tasks to be performed and resources available to complete the project.

NOTES

3. An activity represents an action. It is an effort that consumes time that is needed to complete a part of the overall project.
4. An event is either start or end of an activity.
5. Network is a graphic representation of logically connected activities and events where activities are presented as arrows and events as nodes.
6. Events that are ending of more than one activity are known as merge events and those which are beginning of more than one activity are known as burst events.
7. An activity that consumes neither any resource nor time, but it is there on the network to show a link between events, are known as dummy activity.
8. A dummy activity that is the only activity emanating from an event is a redundant activity which can be eliminated, and this phenomenon is known as redundancy.
9. An activity is critical if delay in its start will cause further delay in completion of the entire project and critical path is the sequence of all such activities in the network.
10. PERT stands for Program Evaluation Review Technique. It is a probabilistic method where activity times are represented by a probability distribution. PERT is used where activities involved in a project are non-deterministic in nature.
11. Smallest time taken to complete an activity assuming that everything goes well is optimistic time. Most likely time is the normal time taken by an activity assuming normal delay. Pessimistic time is the longest time required for an activity to complete assuming everything going wrong.
12. Expected time for an activity in a project is denoted in a PERT chart and is given by the formula as given below:  

$$\text{Expected time} = (\text{Optimistic time} + 4 \times \text{Most likely time} + \text{Pessimistic time})/6$$
13. Variance of an activity is given by  $\sigma^2 = [(\text{Pessimistic time} - \text{Optimistic time})/6]^2$ .
14. Standard normal variable is given by the formula;  $(\text{Scheduled time for project completion} - \text{Normal expected time for the project})/\text{Expected standard deviation for the project}$ . Expected standard deviation for the project is calculated by the square root of sum of variance of all the critical activities.
15. PERT and CPM are used together because they have similarities.
16. Crashing of a project means intentionally reducing the duration of a project by allocating more resources to it. A project can be crashed by crashing its critical activities (because the duration of a project is dependent upon the duration of its critical activities). The use of more workers, better equipment, overtime, etc, would generate higher direct costs for individual activities.
17. The crash time is the shortest time that could be achieved if all efforts (at any reasonable cost) were made to reduce the activity time. The limit beyond which the duration of the activity does not decrease by adding any amount of resources is called the crash time. It is the shortest possible activity time.
18. Resource allocation (also known as resource scheduling) implies the task of allocation of resources to various activities in such a manner that the allocation

is considered as acceptable under the given situation. The task of allocation of resources is important as the final schedule depends upon the quantity of deployment of resources.

## 5.12 QUESTIONS AND EXERCISES

### Short-Answer Questions

1. What is understood by a project?
2. What is dangling in a network? How can it be avoided?
3. Write basic two differences between PERT and CPM.
4. How are time estimates used in PERT and CPM?
5. How many types of float are there?
6. Differentiate between float and slack times.
7. Define critical activity and critical path?
8. The total float of an activity  $i - j$  is 18. The latest and earliest occurrence of events  $i$  and  $j$  are 15, 12 and 22, 10 respectively. Find the free float.
9. What is independent float when the total float of an activity  $i - j$  is 18? Latest and earliest occurrence of events  $i$  and  $j$  are 15, 12 and 22, 10 respectively.
10. What are the limitations of PERT and CPM?
11. Differentiate between crash time and crash cost.
12. Define the significance of resource leveling.

### Long-Answer Questions

1. The following table gives the activities and duration of a construction project.

Activity	1-2	1-3	2-3	2-4	3-4	4-5
Duration (days)	20	25	10	12	6	10

- (i) Draw the network for the project.
- (ii) Find the critical path.

2. A small project consists of 11 activities  $A, B, C, \dots, K$ . The precedence relationship  $A, B$  can start simultaneously. Given  $A < C, D, I; B < G, F; D < G, F; F < H, K; G, H < J; I, J, K < E$ . The duration of the activities are as follows.

Activity	A	B	C	D	E	F	G	H	I	J	K
Duration (Days)	5	3	10	2	8	4	5	6	12	8	9

Draw the network of the project. Summarise the CPM calculations in a tabular form computing total, and free floats of activities and hence determine the critical path.

... and determine the critical path for the given data. Also calculate all the floats involved in CPM.

Jobs	1-2	1-3	2-4	3-4	3-5	4-5	4-6	5-6
Duration	6	5	10	3	4	6	2	9

4. A small maintenance project consists of the following 12 jobs.

Jobs	1-2	2-3	2-4	3-4	3-5	4-6	5-8	6-7	6-10	7-9	8-9	9-10
Duration (Days)	2	7	3	3	5	3	5	8	4	4	1	7

Draw the arrow network of the project. Summarize CPM calculations in a tabular form calculating the three types of floats and hence determine the critical path.

5. Consider the following data for activities in a given project.

Activity	A	B	C	D	E	F
Predecessor	-	A	-	B,C	C	D,E
Time (Days)	5	4	7	3	4	2

Draw the arrow diagram for the project. Compute the earliest and the latest event times. What is the minimum project completion time? List the activities on the critical path.

6. For the following project, determine the critical path and its duration?

Activity	A	B	C	D	E	F	G	H
Predecessors	-	A	A	B	B	D,E	D	C,F,G
Time (Days)	2	4	8	3	2	3	4	8

7. A project has the following time schedule.

Activity	1-2	1-3	1-4	2-5	3-6	3-7	4-6	5-8	6-9	7-8	8-9
Duration (Month)	2	2	1	4	8	5	3	1	5	4	3

Construct the network and compute:

- Total float for each activity.
- Critical path and its duration.

8. The data for a small PERT project is as given below, where  $a$  represents optimistic time,  $m$  the most likely time and  $b$  the pessimistic time. Estimates (in days) of the activities  $A, B, \dots, J, K$ .

Activity	A	B	C	D	E	F	G	H	I	J	K
$a$	3	2	6	2	5	3	3	1	4	1	2
$m$	6	5	12	5	11	6	9	4	19	2	4
$b$	5	14	30	8	17	15	27	7	28	9	12

$A, B, C$  can start simultaneously;  $A \leq D, I; B < G, F; D < G, F; C < E; E < H, K; F < H, K; G, H < J$ .

- Draw the arrow network of the project.
- Calculate the earliest and the latest expected times to each event and find critical path.
- What is the probability that the project will be completed 2 days later than expected?

9. The three estimates for the activities of a project are given below:

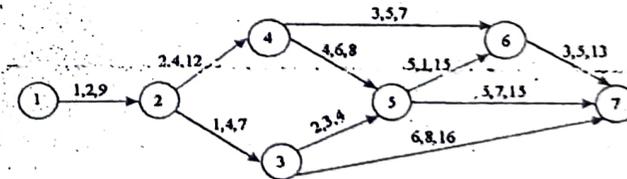
Activity	Estimate Duration (Days)		
	$a$	$m$	$b$
1-2	5	6	7
1-3	1	1	7
1-4	2	4	12
2-5	3	6	15
3-5	1	1	1
4-6	2	2	8
5-6	1	4	7

Draw the project network. Find out the critical path of the project and project duration. What is the probability that the project will be completed at least 5 days earlier than expected?

What is the probability that the project will be completed by 22 days?

10. Consider the network shown in the figure below. The estimate  $t_o$ ,  $t_m$  and  $t_p$  are shown in this order for each of the activities on the top of the arcs denoting the respective activities.

Find the probability of completing the project in 25 days.



11. A project is represented by the network shown below and has the following table:

Task	A	B	C	D	E	F	G	H	I
Least time	5	18	26	16	15	6	7	7	3
Greatest time	10	22	40	20	25	12	12	9	5
Most likely time	8	20	33	18	20	9	10	8	4

Determine the following:

- Expected tasks time and their variance.
- The earliest and the latest expected time to reach each mode.

(iii) The critical path.

(iv) The probability of completing the project within 41.5 weeks.

12. Consider a project having the following activities and their time estimates.

Draw an arrow diagram for the project. Identify the critical path and compute the expected project completion time. What is the probability that the project will require atleast 75 days?

Activity	Predecessor	Days		
		$t_o$	$t_m$	$t_p$
A	-	2	4	6
B	A	8	12	16
C	A	14	16	30
D	B	4	10	16
E	C,B	6	12	18
F	E	6	8	22
G	D	18	18	30
H	F,G	8	14	32

13. Compare PERT and CPM with the help of examples.

### 5.13 FURTHER READING

Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.

Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.

Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.

# UNIT 6 INVENTORY, QUEUING MODELS AND SIMULATION

## Structure

- 6.0 Introduction
- 6.1 Unit Objectives
- 6.2 Queuing Models
  - 6.2.1 Kendall's Notation for Representing Queuing Models
  - 6.2.2 Classification of Queuing Models
- 6.3 Simulation
  - 6.3.1 Simulation of a Queuing System Using Event List
  - 6.3.2 Multiplicative Congruential Algorithm
  - 6.3.3 Inverse Transformation Method
- 6.4 Monte-Carlo Simulation
- 6.5 Inventory Problems
  - 6.5.1 EOQ with Price Breaks
- 6.6 Summary
- 6.7 Key Terms
- 6.8 Answers to 'Check Your Progress'
- 6.9 Questions and Exercises
- 6.10 Further Reading

## 6.0 INTRODUCTION

In this unit, you will understand the basics of queueing theory and its uses in real life. A flow of customers from finite/infinite population towards the service facility forms a 'queue' (waiting line), on account of a lack of capability to serve them all at a time. In the absence of a perfect balance between the service facilities and the customers, waiting time is required either for the service facilities or for the customer's arrival. The arriving unit that requires some services to be performed is called 'customer'. The customer may be persons, machines, vehicles, etc.

Queue (waiting line) stands for the number of customers waiting to be serviced. This does not include the customers being serviced. The process or system that provides services to the customers is termed as service channel or service facility. We find people standing in queue to take some services from some service facilities. Such queues are found at the counters for booking railway tickets, at bus stand, etc. After entering a service centre, if a person is serviced before arrival of another person, no queue is formed. But if persons arrive while service provider is busy serving a customer, they have to wait and a queue or waiting line is formed. Queueing theory carries out study of waiting lines using mathematical models.

You will learn about queueing discipline. When a queue is formed, service is rendered according to some queueing discipline that may be; 'First Come First Serve' (FCFS) or 'First In First Out' (FIFO), 'Last In First Out' (LIFO), 'Service In Random Order' (SIRO) and priority queue. You will also learn about a queueing system that has four essential components; arrival pattern, service pattern, queue discipline and customer behaviour. Arrival pattern tells how customers are arriving for service at

the service facility. Service pattern tells how service is being rendered to customers. Queue discipline tells the order in which customers are given services. A queue may be considered for service either as FCFS, FIFO, SIRO or on priority basis. You will know about customer behaviour which is an important aspect of queuing system. Some customers may leave the queue seeing the long length of waiting line that is formed (balking). There are customers who find it difficult to wait for long standing in the queue and leave the queue (reneging). Sometimes, it is found that customers move from one waiting line to another when there are many service channels (jockeying).

In this unit, you will learn about states of queuing system. A queuing system is in transient state when its operating characteristics are dependent on time. If the system behaviour is independent of time then it is in steady state. You will also learn about notation for describing the model mathematically. A queuing model is completely described in symbolic form as  $(a/b/c): (d/e)$ , where,  $a$  shows probability law for arrival,  $b$  shows law according to which customers are served,  $c$  shows for number of service channels,  $d$  shows capacity of the system (maximum number allowed in the system) and  $e$  shows queuing discipline.

Finally this unit, you will learn also about simulation. It is a representation of reality through the use of a model or any other device which will react like a real one under a given set of conditions. Simulation can be analog or digital. Digital simulation is also known as computer simulation. You will understand the importance and application of simulation techniques. Simulation is the process of designing a mathematical or logical model of a real system and then accomplishing experiments with the model to describe, explain and predict the behavior of the real system. A static simulation model represents a system at a particular point in time whereas a dynamic simulation model represents a system developed over time. Based on these two classifications, a simulation may be classified as deterministic or stochastic. A deterministic simulation model is one that contains no random variables whereas a stochastic simulation model contains one or more random variables. You will also learn about multiplicative congruential algorithm and inverse transformation method. The Monte Carlo simulation model is used by modern management. It uses random number tables to reproduce on paper the operation of any given system under its own working conditions. You will also know about the three models of Monte Carlo simulation used in three different contexts. These are queuing theory, inventory control and production line.

## 6.1 UNIT OBJECTIVES

After going through this unit, you will be able to:

- Describe queuing theory
- Understand queuing discipline
- Know about customer behaviour
- Explain Kendall's notation
- Define queuing system
- Classify various queuing mathematical models

- Solve problems on queuing theory
- Understand the basic concepts of simulation
- Analyse the applications and importance of simulation
- Discuss the simulation of a queuing system using event list
- Describe the significance of multiplicative congruential algorithm and inverse transformation method
- Understand Monte Carlo simulation and its application
- Understand the three context models of Monte Carlo simulation
- Describe the problems related to inventory

## 6.2 QUEUING MODELS

A queuing system can be completely described by:

- (i) The input (arrival pattern).
- (ii) The service mechanism (service pattern).
- (iii) The queue discipline.
- (iv) Customer's behaviour.

### The Input (Arrival Pattern)

Input describes the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random fashion, which is not possible to predict. Thus, the arrival pattern can be described in terms of probabilities, and consequently, the probability distribution for inter-arrival times (the time between two successive arrivals) must be defined. We deal with those queuing systems in which the customers arrive in Poisson fashion. The mean arrival rate is denoted by  $\lambda$ .

### The Service Mechanism

This means, the arrangement of service facility to serve customers. If there is an infinite number of servers, then all the customers are served instantaneously on arrival and there will be no queue.

If the number of servers is finite then the customers are served according to a specific order, with service time as a constant or random variable. Distribution of service time that is important in practice is the *negative exponential distribution*. The mean service rate is denoted by  $m$ .

### The Queue Discipline

It is a rule according to which the customers are selected for service when a queue has been formed. The most common disciplines are:

- First Come First Served (FCFS).
- First In First Out (FIFO).
- Last In First Out (LIFO).
- Selection for Service In Random Order (SIRO).

There are various other disciplines according to which a customer is served in preference over the others. Under priority discipline, the service is of two types, namely pre-emptive and non-pre-emptive. In pre-emptive system, the high priority customers are given service over the low priority customers; in non-pre-emptive system, a customer of low priority is serviced before a customer of high priority. In the case of parallel channels 'fastest server rule' is adopted.

### Customer's Behaviour

The customers generally behave in the following four ways:

- (i) **Balking:** A customer who leaves the queue because the queue is too long and he has no time to wait or does not have sufficient waiting space.
- (ii) **Reneging:** This occurs when a waiting customer leaves the queue due to impatience.
- (iii) **Priorities:** In certain applications some customers are served before others, regardless of their arrival. These customers have priority over others.
- (iv) **Jockeying:** Customers may jockey from one waiting line to another. This is most common in a supermarket.

**Transient and Steady States:** A system is said to be in a *transient state* when its operating characteristics are dependent on time.

A steady state system is the one in which the behaviour of the system is independent of time. Let  $P_n(t)$  denote the probability that there are  $n$  customers in the system, at time  $t$ . Then in steady state,

$$\lim_{t \rightarrow \infty} P_n(t) = p_n \quad (\text{Independent of } t)$$

$$\Rightarrow \frac{dp_n(t)}{dt} = \frac{dp_n}{dt}$$

$$\Rightarrow \lim_{t \rightarrow \infty} p'_n(t) = 0$$

**Traffic Intensity (or Utilization Factor):** An important measure of a simple queue is its traffic intensity given by,

$$\text{Traffic Intensity } \rho = \frac{\text{Mean arrival rate}}{\text{Mean service rate}} = \frac{\lambda}{\mu}$$

The unit of traffic intensity is *Erlang*.

### 6.2.1 Kendall's Notation for Representing Queuing Models

Generally, queuing model may be completely specified in the symbol form  $(a/b/c)$ : ( $d/e$ ) where,

- $a$  = Probability law for the arrival (inter-arrival) time.
- $b$  = Probability law according to which the customers are being served.
- $c$  = Number of channels (or service stations).

$d$  = Capacity of the system, i.e., the maximum number allowed in the system (in service and waiting).

$e$  = Queue discipline.

### 6.2.2 Classification of Queuing Models

The queuing models are classified as follows:

**Model I: (M/M/1): ( $\infty$ /FCFS):** This denotes Poisson arrival (exponential inter-arrival), Poisson departure (exponential service time), Single server, Infinite capacity and First come first served service discipline. The letter  $M$  is used due to Markovian property of exponential process.

**Model II: (M/M/1): (N/FCFS):** In this model, the capacity of the system is limited (finite), say  $N$ . Obviously, the number of arrivals will not exceed the number  $N$  in any case.

**Model III: Multiservice Model (M/M/S):( $\infty$ /FCFS):** This model takes the number of service channels as  $S$ .

**Model IV: (M/M/S): (N/FCFS):** This model is essentially the same as model II, except the maximum number of customers in the system is limited to  $N$ , where, ( $N > S$ ).

**Model I: (M/M/1) ( $\infty$ /FCFS) (Birth and Death Model)**

**To Obtain the Steady State Equations:** The probability that there will be  $n$  units ( $n > 0$ ) in the system at time  $(t + \Delta t)$ , may be expressed as the sum of three independent compound probabilities by using the fundamental properties of probability, Poisson arrivals and exponential service times.

The following are the three cases:

Time (t) No. of units	Arrival	Service	Time (t + Δt) No. of units
$n - 1$	0	0	$n$
$n - 1$	1	0	$n$
$n - 1$	?	?	$n - 1$

Now, by adding the above three independent compound probabilities, we obtain the probability of  $n$  units in the system at time  $(t + \Delta t)$ .

$$P_n(t + \Delta t) = P_n(t) (1 - (\lambda + \mu) \Delta t) + P_{n-1}(t) \lambda \Delta t + P_{n+1}(t) \mu \Delta t + 0(\Delta t)$$

$$\frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{P_n(t + \Delta t) - P_n(t)}{\Delta t} = \lim_{\Delta t \rightarrow 0} \left[ -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t) + \frac{0(\Delta t)}{\Delta t} \right]$$

$$\frac{dP_n(t)}{dt} = -(\lambda + \mu) P_n(t) + \lambda P_{n-1}(t) + \mu P_{n+1}(t)$$

#### Check Your Progress

1. Define a queue.
2. What is FCFS in a queuing discipline?
3. What is SIRO?
4. What are the essential components of a queuing system?
5. What is balking?
6. What reneging?
7. What jockeying?
8. Differentiate between transient and steady states.

where,

$$n > 0$$

$$\left( \because \lim_{\Delta t \rightarrow 0} \frac{0 \Delta(t)}{\Delta t} = 0 \right)$$

In the steady state,

$$P_n(t) \rightarrow 0, P_n(t) = P_n$$

$$0 = -(\lambda + \mu)P_n + \lambda P_{n-1} + \mu P_{n+1} \quad (1)$$

In a similar fashion, the probability that there will be  $n$  units (i.e.,  $n = 0$ ) in the system at time  $(t + \Delta t)$ , will be the sum of the following two independent probabilities.

(i) Probability (that there is no unit in the system at time  $t$  and no arrival in time  $\Delta t$ )

$$= P_0(t) (1 - \lambda \Delta t)$$

(ii) Probability (that there is one unit in the system at time  $t$ , one unit serviced in  $\Delta t$  and no arrival in  $\Delta t$ )

$$= P_1(t) \mu \Delta t (1 - \lambda \Delta t)$$

$$= P_1(t) \mu \Delta t - 0(\Delta t)$$

Adding these two probabilities,

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda \Delta t) + P_1(t) \mu \Delta t + 0(\Delta t)$$

$$\frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) + \frac{0(\Delta t)}{\Delta t}$$

$$\lim_{\Delta t \rightarrow 0} \frac{P_0(t + \Delta t) - P_0(t)}{\Delta t} = -\lambda P_0(t) + \mu P_1(t) \quad \text{for } n = 0$$

$$\frac{dP_0(t)}{dt} = -\lambda P_0(t) + \mu P_1(t)$$

Under steady state, we have,

$$0 = -\lambda P_0 + \mu P_1 \quad (2)$$

Equations (1) and (2) are called *steady state difference equations* for this model.

From (2),

$$P_1 = \frac{\lambda}{\mu} P_0$$

From (1),

$$P_2 = \frac{\lambda}{\mu} P_1 = \left(\frac{\lambda}{\mu}\right)^2 P_0$$

Generally,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0$$

Since,

$$\sum_{n=0}^{\infty} P_n = 1$$

$\Rightarrow$

$$P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 + \dots = 1$$

$$P_0 \left[ 1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2 + \dots \right] = 1$$

i.e.,

$$P_0 \left[ \frac{1}{1 - \frac{\lambda}{\mu}} \right] = 1$$

Since,  $\frac{\lambda}{\mu} < 1$ , sum of infinite G.P. is valid.

$$P_0 = 1 - \frac{\lambda}{\mu} = 1 - \rho$$

Also,

$$P_n = \left(\frac{\lambda}{\mu}\right)^n P_0 = \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right)$$

$$P_n = \rho^n (1 - \rho)$$

### Measures of Model I

1. Expected (average) number of units in the system  $L_S$

$$\begin{aligned} L_S &= \sum_{n=1}^{\infty} n P_n \\ &= \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^n \left(1 - \frac{\lambda}{\mu}\right) \\ &= \left(1 - \frac{\lambda}{\mu}\right) \frac{\lambda}{\mu} \sum_{n=1}^{\infty} n \left(\frac{\lambda}{\mu}\right)^{n-1} \\ &= \left(1 - \frac{\lambda}{\mu}\right) \frac{\lambda}{\mu} \left[ 1 + 2\left(\frac{\lambda}{\mu}\right) + 3\left(\frac{\lambda}{\mu}\right)^2 + \dots \right] \\ &= \left(1 - \frac{\lambda}{\mu}\right) \left(\frac{\lambda}{\mu}\right) \left(1 - \frac{\lambda}{\mu}\right)^{-2} \\ &= \frac{\frac{\lambda}{\mu}}{1 - \frac{\lambda}{\mu}} = \frac{\rho}{1 - \rho}, \quad \rho = \frac{\lambda}{\mu} < 1. \end{aligned}$$

$$L_S = \frac{\rho}{1 - \rho}$$

2. Expected (average) queue length  $L_q$ .

$$L_q = L_S - \frac{\lambda}{\mu} = \frac{\rho^2}{1 - \rho}$$

$$L_q = \frac{\rho^2}{1 - \rho}$$

3. Expected waiting line in the queue,

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)} = \frac{\rho}{1-\rho}$$

4. Expected waiting line in the system,

$$W_s = W_q + \frac{1}{\mu}$$

$$= \frac{\lambda}{\mu(\mu-\lambda)} + \frac{1}{\mu} = \frac{1}{\mu-\lambda}$$

$$W_s = \frac{1}{\mu-\lambda}$$

5. Expected waiting time of a customer who has to wait ( $W/W > 0$ )

$$(W/W > 0) = \frac{1}{\mu-\lambda} = \frac{1}{\mu(1-\rho)}$$

6. Expected length of non-empty queue,

$$(L/L > 0) = \frac{\mu}{\mu-\lambda} = \frac{1}{1-\rho}$$

7. Probability of queue size  $\geq N = \rho^N$

$$= \int_0^{\infty} \rho(\mu-\lambda)e^{-(\mu-\lambda)W} dW$$

8. Probability of waiting time in the queue  $\geq t$

$$= \int_0^{\infty} \lambda(1-\rho)e^{-(\mu-\lambda)W} dW$$

9. Probability (waiting time in the system  $\geq t$ )

$$= \int_0^{\infty} \rho(\mu-\lambda)e^{-(\mu-\lambda)W} dW$$

10. Traffic intensity,  $\rho = \frac{\lambda}{\mu}$

Inter-Relationship between  $L_s, L_q, W_s, W_q$  (Little's Formula)

We know,

$$L_s = \frac{\lambda}{1-\frac{\lambda}{\mu}} = \frac{\lambda}{\mu-\lambda}$$

$$W_s = \frac{1}{\mu-\lambda}$$

$$L_s = \lambda W_s$$

Similarly,  $L_q = \lambda W_q$  hold in general,

$$W_q = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$W_s = \frac{1}{\mu-\lambda}$$

$$W_s - \frac{1}{\mu} = \frac{1}{\mu-\lambda} - \frac{1}{\mu} = \frac{\mu - (\mu-\lambda)}{\mu(\mu-\lambda)} = \frac{\lambda}{\mu(\mu-\lambda)}$$

$$W_q = W_s - \frac{1}{\mu}$$

Multiplying both sides by  $\lambda$ , we have,

$$\lambda W_q = \lambda \left( W_s - \frac{1}{\mu} \right)$$

$$L_q = \lambda W_s - \frac{\lambda}{\mu} = L_s - \frac{\lambda}{\mu}$$

$$L_q = L_s - \frac{\lambda}{\mu}$$

**Example 6.1:** A T.V. mechanic finds that the time spent on his jobs has an exponential distribution with mean 30 minutes, if he repairs sets in the order in which they come in. If the arrival of sets is approximately Poisson with an average rate of 10 per eight-hour day, what is the mechanic's expected idle time each day? How many jobs are ahead of the average set just brought in?

**Solution:**

Here,  $\mu = 1/30, \lambda = \frac{10}{8 \times 60} = \frac{1}{48}$

Expected number of jobs are,

$$L_s = \frac{\lambda}{1 - \frac{\lambda}{\mu}} = \frac{\lambda}{\mu - \lambda}$$

$$= \frac{1}{1/30 - 1/48} = 1 \frac{2}{3} \text{ jobs.}$$

Since the fraction of the time the mechanic is busy equals to  $\frac{\lambda}{\mu}$ , the number of hours for which the repairman remains busy in an eight-hour day,

$$= 8 \left( \frac{\lambda}{\mu} \right) = 8 \times \frac{30}{48} = 5 \text{ hours}$$

Therefore, the time for which the mechanic remains idle in an eight-hour day =  $(8 - 5)$  hours = 3 hours.

**Example 6.2:** At what average rate must a clerk at a supermarket work, in order to insure a probability of 0.90 that the customers will not have to wait longer than 12 minutes? It is assumed that there is only one counter, to which customers

arrive in a Poisson fashion at an average rate of 15 per hour. The length of service by the clerk has an exponential distribution.

Solution:

Here,  $\lambda = \frac{15}{60} = \frac{1}{4}$  customer/minute  $\mu = ?$

Prob. (waiting time  $\geq 12$ ) =  $1 - 0.9 = 0.10$

$$\int_0^{\infty} \lambda(1-\lambda)e^{-(\mu-\lambda)\omega} d\omega$$

$$\int_{12}^{\infty} \lambda \left(1 - \frac{\mu}{\lambda}\right) e^{-(\mu-\lambda)\omega} d\omega = 0.1$$

$$\lambda \left(1 - \frac{\mu}{\lambda}\right) \left(\frac{e^{-(\mu-\lambda)\omega}}{-(\mu-\lambda)}\right)_{12}^{\infty} = 0.1$$

$$\frac{\lambda}{\mu} (e^{-12(\mu-\lambda)}) = 0.10$$

$$e^{(3-12\mu)} = 0.4\mu$$

$$\frac{1}{\mu} = 2.48 \text{ minutes per service.}$$

**Example 6.3:** Arrivals at a telephone booth are considered to be Poisson, with an average time of 10 minutes between one arrival and the next. The length of a phone call is assumed to be distributed exponentially with mean three minutes.

- (i) What is the probability that a person arriving at the booth will have to wait?
- (ii) What is the average length of the queue that forms from time to time?
- (iii) The telephone department will install a second booth when convinced that an arrival would have to wait at least three minutes for the phone. By how much time must the flow of arrivals be increased in order to justify a second booth?

Solution:

Given,  $\lambda = 1/10, \mu = 1/3$

(i) Probability ( $w > 0$ ) =  $1 - P_0 = \frac{\lambda}{\mu} = \frac{1}{10} \times \frac{3}{1} = 3/10 = 0.3$

(ii)  $(L/L > 0) = \frac{\mu}{\mu - \lambda} = \frac{1/3}{1/3 - 1/10} = 1.43$  persons

(iii)  $W_q = \frac{\lambda}{\mu(\mu - \lambda)}$

Since,  $W_q = 3, \mu = \frac{1}{3}, \lambda = \lambda'$  for second booth,

$$3 = \frac{\lambda'}{\frac{1}{3}(\frac{1}{3} - \lambda')} \Rightarrow \lambda' = 0.16$$

Hence, increase in the arrival rate =  $0.16 - 0.10 = 0.06$  arrival per minute.

**Example 6.4:** As in Example 6.3, in a telephone booth with Poisson arrivals spaced 10 minutes apart on the average and exponential call length averaging three minutes.

- (i) What is the probability that an arrival will have to wait for more than 10 minutes before the phone becomes free?
- (ii) What is the probability that it will take him more than 10 minutes in total to wait for the phone and complete his call?
- (iii) Estimate the fraction of a day that the phone will be in use.
- (iv) Find the average number of units in the system.

Solution: Given,

$n\lambda = 0.1$  arrival/minute  
 $\mu = 0.33$  service/minute

(i) Probability (waiting time  $\geq 10$ ) =  $\int_{10}^{\infty} \left(1 - \frac{\lambda}{\mu}\right) \lambda e^{-(\mu-\lambda)W} dW$

$$= -\frac{\lambda}{\mu} \left(e^{-(\mu-\lambda)W}\right)_{10}^{\infty}$$

$$= 0.3 e^{-2.3} = 0.03$$

(ii) Probability (waiting time in the system  $\geq 10$ )

$$= \int_{10}^{\infty} (\mu - \lambda) e^{-(\mu-\lambda)W} dW$$

$$= e^{-10(\mu-\lambda)} = e^{-2.3} = 10$$

(iii) The fraction of a day that the phone will be busy = traffic intensity

$$\rho = \frac{\lambda}{\mu} = 0.3.$$

(iv) Average number of units in the system,

$$L_S = \frac{\lambda}{\mu - \lambda} = \frac{1}{\frac{1}{3} - \frac{1}{10}} = 3/7 = 0.43 \text{ customer.}$$

**Example 6.5:** Customers arrive at a one-window drive-in bank according to Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean five minutes. The space in front of the window including that for the serviced car can accommodate a maximum of three cars. Others can wait outside this space.

- (i) What is the probability that an arriving customer can drive directly to the space in front of the window?
- (ii) What is the probability that an arriving customer will have to wait outside the indicated space?
- (iii) How long is an arriving customer expected to wait before starting service?

Solution: Given,

$\lambda = 10$  per hour

$\mu = \frac{1}{5} \times 60 = 12$  per hour

$\rho = \frac{\lambda}{\mu} = \frac{10}{12}$

- (i) The probability that an arriving customer can drive directly to the space in front of the window,

$$\begin{aligned}
 P_0 + P_1 + P_2 &= P_0 + \frac{\lambda}{\mu} P_0 + \left(\frac{\lambda}{\mu}\right)^2 P_0 \\
 &= P_0 \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right) \\
 &= \left(1 - \frac{\lambda}{\mu}\right) \left(1 + \frac{\lambda}{\mu} + \left(\frac{\lambda}{\mu}\right)^2\right) \quad \because P_0 = 1 - \frac{\lambda}{\mu} \\
 &= \left(1 - \frac{10}{12}\right) \left(1 + \frac{10}{12} + \frac{100}{144}\right) = 0.42
 \end{aligned}$$

(ii) Probability that an arriving customer will have to wait outside the indicated space,

$$S = 1 - 0.42 = 0.58$$

(iii) Average waiting time of a customer in a queue,

$$\begin{aligned}
 &= \frac{\lambda}{\mu} \frac{1}{\mu - \lambda} = \frac{10}{12} \left(\frac{1}{12 - 10}\right) = \frac{5}{12} \\
 &= 0.417 \text{ hours.}
 \end{aligned}$$

**Example 6.6:** In a supermarket, the average arrival rate of customers is 10 every 30 minutes, following Poisson process. The average time taken by a cashier to list and calculate the customer's purchase is two and a half minutes following exponential distribution. What is the probability that the queue length exceeds six?

What is the expected time spent by a customer in the system?

**Solution:**

$$\lambda = \frac{10}{30} \text{ per minute}$$

$$\mu = \frac{1}{2.5} \text{ per minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{\frac{1}{3}}{1/2.5} = 0.8333$$

(i) The probability of queue size  $> 6 = \rho^6$

$$\text{Expected waiting time } W_s = \frac{1}{\mu - \lambda} = (0.8333)^6 = 0.3348.$$

(ii) 
$$W_s = \frac{L_s}{\lambda} = \frac{1 - \rho}{\lambda} = \frac{0.833}{1 - 0.8333} \times 3 = 14.96 \text{ minutes.}$$

**Example 6.7:** On an average, 96 patients per 24-hour day require the service of an emergency clinic. Also, on an average a patient requires 10 minutes of active attention. Assume that the facility can handle only one emergency at a time. Suppose that it costs the clinic ₹ 100 per patient treated, to obtain an average servicing time of 10 minutes and thus, each minute of decrease in this average time would cost ₹ 10 per patient treated. How much would have to be budgeted

by the clinic to decrease the average size of the queue from  $1\frac{1}{3}$  patients to  $\frac{1}{2}$  patients?

**Solution:**

Given,

$$\lambda = \frac{96}{24 \times 60} = \frac{1}{15} \text{ patient/minute}$$

$$\mu = \frac{1}{10} \text{ patient/minute}$$

Average number of patients in the queue,

$$L_q = \frac{\lambda}{\mu} \frac{\lambda}{\lambda - \mu} = \frac{\left(\frac{\lambda}{\mu}\right)^2}{1 - \frac{\lambda}{\mu}}$$

$$= \frac{\left(\frac{1}{15}\right)^2}{\left(\frac{1}{10} - \frac{1}{15}\right) \frac{1}{10}} = 1\frac{1}{3} \text{ patients}$$

But,

$$L_q = 1\frac{1}{3} \text{ is reduced to } L_q' = 1/2$$

$\therefore$  Substituting  $L_q' = 1/2$ ,  $\lambda' = \lambda = \frac{1}{15}$  in the formula

$$L_q' = \frac{\lambda'^2}{\mu'(\mu' - \lambda')}$$

$$\frac{1}{2} = \frac{\left(\frac{1}{15}\right)^2}{\mu'(\mu' - 1/15)} \Rightarrow \mu' = 2/15 \text{ patients/minute}$$

Hence, the average rate of treatment required is,  $\frac{1}{\mu'} = 7.5$  minutes. Decrease in time required by each patient

$$= 10 - \frac{15}{2} = \frac{5}{2} \text{ minutes}$$

$\therefore$  The budget required for each patient

$$= 100 + \frac{5}{2} \times 10 = ₹ 125$$

So, in order to get the required size of the queue, the budget should be increased from ₹ 100 to ₹ 125 per patient.

**Example 6.8:** In a public telephone booth, the arrivals on an average are 15 per hour. A call on an average takes three minutes. If there is just one phone, find (i) the expected number of callers in the booth at any time (ii) the proportion of the time, the booth is expected to be idle?

**Solution:**

Given,

$$\lambda = 15 \text{ per hour}$$

$$\mu = \frac{1}{3} \times 60 \text{ per hour}$$

(i) Expected length of the non-empty queue

$$= \frac{\mu}{\mu - \lambda} = \frac{20}{20 - 15} = 4$$

(ii) The service is busy =  $\frac{\lambda}{\mu} = \frac{15}{20} = \frac{3}{4}$

∴ the booth is expected to be idle for  $1 - \frac{3}{4} = \frac{1}{4}$  hours = 15 minutes.

**Example 6.9:** In a railway marshalling yard, goods trains arrive at a rate of 30 trains per day. Assuming that inter-arrival time and service time distribution follows an exponential distribution with an average of 30 minutes, calculate the following.

(i) The mean queue size.

(ii) The probability that queue size exceeds 10.

(iii) If the input of the train increases to an average of 33 per day, what will be the changes in (i) and (ii)?

**Solution:**

Given,

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48} \text{ trains/minute}$$

$$\mu = \frac{1}{30} \text{ trains/minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{30}{48} = \frac{5}{8}$$

$$(i) L_s = \frac{\rho}{1 - \rho} = \frac{5/8}{1 - 5/8} = \frac{5/8}{3/8} = \frac{5}{3} = 1.66 \text{ trains} = 2 \text{ trains (app.)}$$

$$(ii) P(\geq 10) = (0.75)^{10} = 0.056$$

(iii) When the input increases to 33 trains per day,

$$\text{we have, } \lambda = \frac{33}{60 \times 24} = \frac{11}{480}, \mu = \frac{1}{30} \text{ trains/min}$$

$$\Rightarrow L_s = \frac{\rho}{1 - \rho} \text{ where, } \rho = \frac{\lambda}{\mu} = \frac{11}{16}$$

$$L_s = \frac{11/16}{5/16} = \frac{11}{5} = 2.1 \text{ trains}$$

$$\text{also, } \rho(\geq 10) = \rho^{10} = \left(\frac{11}{16}\right)^{10} = 0.1460.$$

### Model II (M/M/1): (N/FCFS)

This model differs from model I in the sense that, the maximum number of customers in the system is limited to  $N$ . Therefore, the difference equations of model I are valid for this model as long as  $n < N$ . Arrivals will not exceed  $N$  in any case. The various measures of this model are,

$$1. P_0 = \frac{1 - \rho}{1 - \rho^{N+1}} \text{ where, } \rho = \frac{\lambda}{\mu} \left( \frac{\lambda}{\mu} > 1 \text{ is allowed} \right)$$

$$2. P_N = \frac{1 - \rho}{1 - \rho^{N+1}} \rho^N, \text{ for } n = 0, 1, 2, \dots, N$$

$$3. L_q = \frac{\lambda}{\mu}$$

$$4. L_s = \rho_0 \sum_{n=0}^N n \rho^n$$

$$5. L_q = L_s - \frac{\lambda}{\mu}$$

$$6. W_s = L_s / \lambda$$

$$7. W_q = L_q / \lambda$$

**Example 6.10:** In a railway marshalling yard, goods trains arrive at the rate of 30 trains per day. Assume that the inter-arrival time follows an exponential distribution and the service time is also to be assumed as exponential with mean of 36 minutes. Calculate,

(i) The probability that the yard is empty.

(ii) The average queue length, assuming that the line capacity of the yard is nine trains.

**Solution:** For model II

Given,

$$\lambda = \frac{30}{60 \times 24} = \frac{1}{48}, \mu = \frac{1}{36} \text{ trains per minute}$$

$$\rho = \frac{\lambda}{\mu} = \frac{36}{48} = 0.75$$

(i) The probability that the queue is empty is given by,

$$P_0 = \frac{1 - \rho}{1 - \rho^{N+1}} \text{ where } N = 9$$

$$= \frac{1 - 0.75}{1 - (0.75)^{10}} = \frac{0.25}{0.90} = 0.28.$$

(ii) Average queue length is given by,

$$L_s = \frac{1 - \rho}{1 - \rho^{N+1}} \sum_{n=0}^N n \rho^n$$

$$= \frac{1 - 0.75}{1 - (0.75)^{10}} \sum_{n=0}^9 n (0.75)^n$$

$$= 0.28 \times 9.58 = 3 \text{ trains.}$$

**Example 6.11:** A barber shop has space to accommodate only 10 customers. He can serve only one person at a time. If a customer comes to his shop and finds it full, he goes to the next shop. Customers randomly arrive at an average rate  $\lambda = 10$  per hour and the barber's service time is negative exponential with an average of  $1/\mu = 5$  minutes per customer. Find  $P_0$  and  $P_n$ .

**Solution:**

Given,

$$N = 10, \lambda = \frac{10}{60}, \mu = \frac{1}{5}$$

$$\rho = \frac{\lambda}{\mu} = \frac{5}{6}$$

$$P_0 = \frac{1-\rho}{1-\rho^{11}} = \frac{1-5/6}{1-(5/6)^{11}}$$

$$= \frac{0.1667}{0.8655} = 0.1926$$

$$P_n = \left( \frac{1-\rho}{1-\rho^{n+1}} \right) \rho^n$$

$$= (0.1926) \times \left( \frac{5}{6} \right)^n \quad [n = 0, 1, 2, \dots, 10]$$

**Example 6.12:** A car park contains 5 cars. The arrival of cars is Poisson, at a mean rate of 10 per hour. The length of time each car spends in the car park is exponential distribution with mean of 0.5 hours. How many cars are in the car park on an average?

**Solution:**

Given,  $N = 5, \lambda = \frac{10}{60}, \mu = \frac{0.5}{60} = \frac{1}{2 \times 60}$

$$\rho = \frac{\lambda}{\mu} = 20$$

$$P_0 = \left( \frac{1-\rho}{1-\rho^{N+1}} \right) = \frac{1-20}{1-20^6} = \frac{-19}{-63999999}$$

$$= 2.9692 \times 10^{-7}$$

$$L_s = P_0 \sum_{n=0}^N n \rho^n = (2.9692 \times 10^{-7}) \sum_{n=0}^5 n (2.9692 \times 10^{-7})^n$$

$$= 4 \text{ (approximately).}$$

**Example 6.13:** Assuming for a period of two hours in a day (8-10 am), trains arrive at the yard every 20 minutes, then calculate for this period.

- (i) The probability that the yard is empty
- (ii) Average queue length, assuming that the capacity of the yard is 4 trains only.

**Solution:**

Given,  $\rho = \frac{36}{20} = 1.8, \quad N = 4$

(i)  $P_0 = \frac{\rho-1}{\rho^5-1} = 0.04$

(ii) Average queue size

$$= \rho_0 \sum_{n=0}^4 n \cdot \rho^n$$

$$= 0.04(\rho + 2\rho^2 + 3\rho^3 + 4\rho^4) = 0.04 (67.77)$$

$$= 2.9 = 3 \text{ trains.}$$

**Model III: (Multiservice Model) (M/M/S): (∞/FCFS)**

When there are  $n$  units in the system, difference equations may be obtained in the following two situations.

(i) If  $n \leq s$ , all the customers may be served simultaneously. There will be no queue.  $(s-n)$  number of servers may remain idle and then,

$$\mu_n = n\mu, \quad n = 0, 1, 2, \dots, S;$$

(ii) If  $n \geq s$ , all the servers are busy, and the maximum number of customers waiting in queue will be  $(n-s)$ , then,  $\mu_n = s\mu$

Also,  $\lambda_n = \lambda \quad [n = 0, 1, 2, \dots]$

The steady state difference equations are,

$$P_0(t + \Delta t) = P_0(t)(1 - \lambda\Delta t) + P_1(t)\mu\Delta t + O(\Delta t)$$

for  $n = 0$ .

$$P_n(t + \Delta t) = P_n(t)[1 - (\lambda + n\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)(n+1)\mu\Delta t + O(\Delta t) \text{ for } n = 1, 2, \dots, S-1$$

$$P_n(t + \Delta t) = P_n(t)[1 - (\lambda + S\mu)\Delta t] + P_{n-1}(t)\lambda\Delta t + P_{n+1}(t)S\mu\Delta t + O(\Delta t) \text{ for } n = S, S+1, S+2, \dots$$

Now, dividing these equations by  $\Delta t$  and taking limit as  $\Delta t \rightarrow 0$ , the difference equations are,

$$P_0^1(t) = -\lambda P_0(t) + \mu P_1(t) \text{ for } n = 0$$

$$P_n^1(t) = -(\lambda + n\mu)P_n(t) + \lambda P_{n-1}(t) + (n+1)\mu P_{n+1}(t)$$

for,  $n = 1, 2, \dots, S-1$ .

$$P_n^1(t) = -(\lambda + S\mu)P_n(t) + \lambda P_{n-1}(t) + S\mu P_{n+1}(t)$$

for,  $n = S, S+1, S+2, \dots$

Considering the case of steady state, i.e., when  $t \rightarrow \infty, P_n(t) \rightarrow P_n$  and hence,  $P_n^1(t) \rightarrow 0$  for all  $n$ , above equations become,

$$0 = -\lambda P_0 + \mu P_1 \text{ for } n = 0 \quad (1)$$

$$0 = -(\lambda + n\mu)P_n + \lambda P_{n-1} + (n+1)\mu P_{n+1} \quad (2)$$

for,  $1 \leq n \leq S-1$

$$0 = -(\lambda + S\mu)P_n + \lambda P_{n-1} + S\mu P_{n+1} \text{ for } n \leq S \quad (3)$$

Here,

$P_0 = P_0$  initially,

$P_1 = \frac{\lambda}{\mu} P_0$  from (1)

$P_2 = \frac{\lambda}{2\mu} P_1 = \frac{\lambda^2}{2!\mu^2} P_0$  (Put  $n = 1$  in (2))

$P_3 = \frac{\lambda}{3\mu} P_2 = \frac{\lambda^3}{3!\mu^3} P_0$  (Put  $n = 2$  in (2))

In general,

$$P_n = \frac{\lambda}{n\mu} P_{n-1} = \frac{1}{n!} \left( \frac{\lambda}{\mu} \right)^n P_0$$

$1 \leq n \leq S$

$$P_S = \frac{\lambda}{S\mu} P_{S-1} = \frac{1}{S!} \left(\frac{\lambda}{\mu}\right)^S \cdot P_0$$

$$P_{S+1} = \frac{\lambda}{S\mu} P_S = \frac{1}{S! S!} \left(\frac{\lambda}{\mu}\right)^{S+1} \cdot P_0$$

$$P_{S+2} = \frac{\lambda}{S\mu} P_{S+1} = \frac{1}{S^2 S!} \left(\frac{\lambda}{\mu}\right)^{S+2} \cdot P_0$$

In general,

$$P_n = P_{S+(n-S)} = \frac{1}{S^{n-S} S!} \left(\frac{\lambda}{\mu}\right)^n P_0 \quad n \geq S$$

Now, find  $P_n$  using the fact,

$$\sum_{n=0}^{\infty} P_n = 1$$

i.e., 
$$\sum_{n=0}^{S-1} P_n + \sum_{n=S}^{\infty} P_n = 1$$

This gives the steady state distribution of arrivals ( $n$ ) as,

$$P_n = \begin{cases} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0, & n = 0, 1, 2, \dots, S-1 \\ \frac{1}{S! S^{n-S}} \left(\frac{\lambda}{\mu}\right)^n P_0, & \text{if } n = S, S+1, \dots \end{cases}$$

### Measures of Model III

1. Length of the queue,

$$L_q = P_S \frac{\rho}{(1-\rho)^2}$$

where, 
$$P_S = \frac{\left(\frac{\lambda}{\mu}\right)^S P_0}{S!}$$

2. Length of the system,

$$L_S = \frac{\lambda}{\mu} + L_q$$

3. Waiting time in the queue,

$$W_q = \frac{L_q}{\lambda}$$

4. Waiting time in the system,

$$W_S = \frac{L_S}{\lambda}$$

5. The mean number of individuals who actually wait is given by,

$$L(L > 0) = \frac{1}{1-\rho}$$

6. The mean waiting time in the queue for those who actually wait is given by  $W(W > 0)$ ,

$$= \frac{1}{S\mu - \lambda}$$

7. Probability  $P(W > 0) = \frac{P_S}{1-\rho}$

8. Probability that there will be someone waiting =  $\frac{P_S \rho}{1-\rho}$

9. Average number of idle servers =  $S - (\text{Average number of customers served})$

10. Efficiency of M/M/S model =  $\frac{\text{Average number of customers served}}{\text{Total number of customers served}}$

**Example 6.14:** A supermarket has two girls ringing up sales at the counters. If the service time for each customer is exponential with mean four minutes and if people arrive in a Poisson fashion at the counter, at the rate of 10 per hour, then calculate,

- The probability of having to wait for service.
- The expected percentage of idle time for each girl.
- If a customer has to wait, find the expected length of his waiting time.

**Solution:**

(i) Probability of waiting time for service is,

$$P(W > 0) = \frac{\left(\frac{\lambda}{\mu}\right)^S P_0}{S!(1-\rho)}$$

$$\lambda = \frac{1}{6}, \mu = 1/4, S = 2$$

$$\rho = \frac{\lambda}{\mu S} = \frac{1}{3}$$

$$P_0 = \left[ \sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!(1-\rho)} \right]^{-1}$$

$$= \left[ \sum_{n=0}^1 \frac{\left(2 \cdot \frac{1}{3}\right)^n}{n!} + \frac{\left(2 \cdot \frac{1}{3}\right)^2}{2!(1-1/3)} \right]^{-1}$$

$$= 1 + 2/3 + \left[ \frac{4}{2 \times 2/3} \right]^{-1} = \frac{1}{2}$$

Thus, probability  $(W > 0) = \frac{(4/6)^2 \cdot \frac{1}{2}}{2!(1-1/3)} = 1/6$

(ii) The fraction of the time the service remains busy,

(i.e., traffic intensity) is given by,  $\rho = \frac{\lambda}{S\mu} = \frac{1}{3}$

∴ The fraction of the time the service remains idle is,

$$\left(1 - \frac{1}{3}\right) = \frac{2}{3} = 67\% \text{ (approximately)}$$

$$(iii) (W/W > 0) = \frac{1}{1-\rho} \cdot \frac{1}{S\mu} = \frac{1}{1-\frac{1}{3}} \cdot \frac{1}{2 \times \frac{1}{4}} = 3 \text{ minutes.}$$

**Example 6.15:** A petrol station has two pumps. The service time follows the exponential distribution with mean four minutes and cars arrive for service in a Poisson process at the rate of 10 cars per hour. Find the probability that a customer has to wait for service. What proportion of time do the pumps remain idle?

**Solution:**

**Given,**

$$S = 2, \lambda = 10 \text{ per hour}$$

$$\mu = \frac{1}{4} \text{ per minute} = \frac{60}{4} = 15 \text{ per hour}$$

$$\rho = \frac{\lambda}{S\mu} = \frac{10}{2 \times 15} = \frac{1}{3}$$

$$(i) \text{ Probability } (W > 0) = \frac{P_S}{1-\rho}, P_S = \frac{\left(\frac{\lambda}{\mu}\right)^S}{S!} P_0$$

where,

$$\begin{aligned} P_0 &= \left[ \sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!(1-\rho)} \right]^{-1} \\ &= \left[ \sum_{n=0}^1 \frac{(\rho S)^n}{n!} + \frac{(\rho S)^2}{2!(1-\rho)} \right]^{-1} \\ &= \left[ 1 + \frac{(\rho S)^1}{1!} + \frac{(\rho S)^2}{2!(1-\rho)} \right]^{-1} \\ &= \left[ 1 + \frac{1}{3} \times 2 + \frac{\left(\frac{1}{3} \cdot 2\right)^2}{2! \left(1 - \frac{1}{3}\right)} \right]^{-1} = \frac{1}{2} \end{aligned}$$

$$\begin{aligned} P(W > 0) &= \frac{P_S}{1-\rho} = \frac{1}{1-\rho} \cdot \frac{\left(\frac{\lambda}{\mu}\right)^S}{S!} P_0 \\ &= \frac{1}{1 - \frac{1}{3}} \left[ \frac{\left(\frac{2}{3}\right)^2 \cdot \frac{1}{2!}}{2!} \right] = \frac{1}{9} \times \frac{3}{2} = 0.167 \text{ (approximately).} \end{aligned}$$

$$(ii) \text{ The duration of time for which the pumps are busy} = \frac{\lambda}{S\mu} = \frac{1}{3}$$

∴ The duration of time for which the pumps remain idle

$$= 1 - \frac{1}{3} = \frac{2}{3} = 67\% \text{ (approximately)}$$

NOTE

**Example 6.16:** A telephone exchange has two long distance operators. The telephone company finds that during the peak load, long-distance calls arrive in a Poisson fashion at an average rate of 15 per hour. The length of service on these calls is approximately distributed with mean length five minutes.

(i) What is the probability that a subscriber will have to wait for his long distance call during the peak hours of the day?

(ii) If the subscribers will wait and be serviced in turn, what is the expected waiting time?

Establish the formula used.

**Solution:**

$$\text{Here, } \lambda = \frac{15}{60} = \frac{1}{4}, \mu = \frac{1}{5}, S = 2$$

$$\rho = \frac{\lambda}{\mu S} = \frac{5}{8}$$

$$\begin{aligned} \text{First, calculate } P_0 &= \left[ \sum_{n=0}^{S-1} \frac{(S\rho)^n}{n!} + \frac{(S\rho)^S}{S!(1-\rho)} \right]^{-1} \\ &= \left[ \sum_{n=0}^1 \frac{(5/4)^n}{n!} + \frac{(5/4)^2}{2!(1-5/8)} \right]^{-1} \\ &= \frac{1}{1 + 5/4 + (5/4)^2 \cdot \frac{1}{2} \cdot 8/3} = 3/13 \end{aligned}$$

$$(i) \quad P(W > 0) = \frac{\left(\frac{\lambda}{\mu}\right)^S}{S!(1-\rho)} \cdot P_0 = \frac{\left(\frac{5}{4}\right)^2 \cdot 3/13}{2!(1-5/8)} = 25/52 = 0.48$$

$$\begin{aligned} (ii) \quad W_q &= L_q / \lambda \\ &= \frac{1}{\lambda} \cdot \frac{\rho(S\rho)^S}{S!(1-\rho)^2} P_0 \\ &= 4 \cdot \frac{5/8 \cdot (5/4)^2}{2!(1-5/8)^2} \cdot \frac{3}{13} = \frac{125}{39} = 3.2 \text{ minutes.} \end{aligned}$$

**Example 6.17:** Four counters are being run on the frontier of a country to check the passports and necessary papers of the tourists. The tourists choose any counter at random. If the arrival at the frontier is Poisson at the rate  $\lambda$  and the service time is exponential with parameter  $\lambda/2$ , what is the steady state average queue at each counter?

**Solution:**

$$\text{Here, } S = 4, \lambda = \lambda, \mu = \lambda/2$$

$$\rho = \frac{\lambda}{\mu S} = \frac{1}{2}$$

$$P_0 = \left[ \sum_{n=0}^3 \frac{2^n}{n!} + \frac{4^4}{4! \left(\frac{1}{2}\right)^4} \right]^{-1}$$

$$= \left[ \frac{2^0}{0!} + \frac{2^1}{1!} + \frac{2^2}{2!} + \frac{2^3}{3!} + \frac{256}{24} \times \frac{1}{8} \right]^{-1}$$

$$= [1 + 2 + 2 + 8/6 + 4/3]^{-1} = [5 + 8/3]^{-1} = 3/23$$

But, the expected queue length  $\lambda q$  is,

$$= \frac{(\lambda/\mu)^S \rho}{S! (1-\rho^2)} P_0 = \frac{2^4}{4!} \cdot \frac{1}{2} \cdot \frac{3}{23} = 4/23.$$

### Model IV (M/M/S): (M/FCFS)

This model is essentially the same as Model III, except that the maximum number of customers in the system is limited to  $N$ , where  $N > S$  ( $S$  = Number of channels).

$$\lambda_n = \begin{cases} \lambda, & 0 \leq n \leq N \\ 0, & n \geq N \end{cases}$$

$$\mu_n = \begin{cases} n\mu, & 0 \leq n \leq S \\ S\mu, & S \leq n \leq N \end{cases}$$

$$P_n = \begin{cases} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n P_0, & 0 \leq n \leq S \\ \frac{1}{S^{n-S} S!} \left(\frac{\lambda}{\mu}\right)^n P_0, & S \leq n \leq N \end{cases}$$

$$P_0 = \left[ \sum_{n=0}^{S-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=S}^N \frac{1}{S^{n-S} S!} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$L_q = \frac{(S\rho)^C \rho}{S!(1-\rho)^2} [1 - \rho^{N-S+1} - (1-\rho)(N-S+1)\rho^{N-S}] P_0$$

$$L_S = L_q + S - P_0 \sum_{n=0}^{S-1} \frac{(S-n)(S\rho)^n}{n!}$$

$$W_S = \frac{L_S}{\lambda'}, \text{ where } \lambda' = \lambda(1 - P_N)$$

$$W_q = W_S - \frac{1}{\mu}$$

**Example 6.18:** A barber shop has two barbers and three chairs for customers. Assume that the customers arrive in a Poisson fashion at a rate of five per hour and each barber services customers according to an exponential distribution with mean 15 minutes. Further, if a customer arrives and there are no empty chairs in the shop, he will leave. What is the expected number of customers in the shop?

**Solution:**

Here,  $S = 2, N = 3, \lambda = \frac{5}{60} = \frac{1}{12}$  customer/min

$$\mu = \frac{1}{15} / \text{min}$$

$$P_0 = \left[ \sum_{n=0}^{2-1} \frac{1}{n!} \left(\frac{\lambda}{\mu}\right)^n + \sum_{n=2}^3 \frac{1}{2^{n-2} 2!} \left(\frac{\lambda}{\mu}\right)^n \right]^{-1}$$

$$= \left[ 1 + 1 \cdot \frac{5}{4} + \frac{1}{2!} \left(\frac{5}{4}\right)^2 + \frac{1}{2 \cdot 2!} \left(\frac{5}{4}\right)^3 \right]^{-1}$$

$$= \left[ 1 + \frac{5}{4} + \frac{1}{2} (5/4)^2 + \frac{1}{4} (5/4)^3 \right]^{-1}$$

$$= \frac{256}{901} = 0.28$$

$$P_n = \begin{cases} \frac{1}{n!} (5/4)^n \times 0.28 & 0 \leq n < 2 \\ \frac{1}{2^{n-2} 2!} (5/4)^n \times 0.28 & 2 \leq n \leq 3 \end{cases}$$

$$= \begin{cases} \frac{1}{n!} (1.25)^n \times 0.28 & 0 \leq n < 2 \\ \frac{1}{2^{n-2} 2!} (1.25)^n \times 0.28 & 2 \leq n \leq 3 \end{cases}$$

$$L_S = L_q + S - P_0 \sum_{n=0}^{S-1} \frac{(S-n) \left(\frac{\lambda}{\mu}\right)^n}{n!}$$

$$= \sum_{n=2}^3 (n-2) P_n + 2 - P_0 \sum_{n=0}^{2-1} \frac{(2-n)(1.25)^n}{n!}$$

$$= P_3 + 2 - 3.2 P_0$$

$$= \left[ \frac{1}{2 \cdot 2!} (1.25)^3 \times 0.28 \right] + 2 - 3.2 \times 0.28 = 1.226 \text{ customers.}$$

## 6.3 SIMULATION

In operations research, simulation is the most widely used flexible modelling approach. This approach is used to model the behaviour of individual components within the system with the help of random sampling technique for generating pragmatic variability. The simulation process includes problem definition, conceptual modelling, model coding, model verification and validation, experimentation and analysis of results, and solution implementation.

### Types of Simulation

Simulation is mainly of two types:

- (i) Analog (environmental) simulation.
- (ii) Computer (system) simulation.

Check Your

9. What is a queueing system?
10. Define a service system.
11. What is an intensity queueing system?
12. What is a model in which models are described?
13. What is a model by a model (M/M/S) (M/FCFS)?
14. Write an mathematical model for FCFS queueing system.

Some examples of simulation models are given below:

- (i) Testing an aircraft model in a wind tunnel.
- (ii) Children cycling in a park with various signals and crossings—to model a traffic system.
- (iii) Planetarium.

To determine the behaviour of a real system in actual environment, a number of experiments are performed on simulated models either in the laboratories or on the computer itself.

### Random Variable

The random variable is a real-valued function, defined over a sample space associated with the outcome of a *conceptual* chance experiment. Random variables are classified according to their probability density function.

**Random Number:** It refers to a uniform random variable or a numerical value assigned to a random variable, following uniform probability density function. In other words, it is a number in a sequence of numbers, whose probability of occurrence is the same as that of any other number in that sequence.

**Pseudo-Random Numbers:** Random numbers are called *pseudo-random numbers* when they are generated by some deterministic process, but have already qualified the pre-determined statistical test for randomness.

### Monte Carlo Technique or Monte Carlo Simulation

The Monte Carlo method is a simulation technique in which statistical distribution functions are created by using a series of random numbers. The method is generally used to solve the problems that cannot be adequately represented by the mathematical models or where the solution of the model cannot be arrived at, by analytical method.

Monte Carlo simulation yields a solution very close to the optimal solution, but not necessarily the exact solution. The Monte Carlo simulation procedure can be summarized in the following six steps.

#### Step 1 Clearly define the problem.

- (a) Identify the objectives of the problem.
- (b) Identify the main factors which have the greatest effect on the objectives of the problem.

#### Step 2 Construct an appropriate model.

- (a) Specify the variables and parameters of the model.
- (b) State the conditions under which the experiment is to be performed.
- (c) Define the relationship between the variables and parameters.

#### Step 3 Prepare the model for experimentation.

- (a) Define the starting conditions for the simulation.
- (b) Specify the number of runs of simulation to be made.

#### Step 4 Using Steps 1 to 3, experiment with the model.

- (a) Define a coding system that will correlate the factors defined in Step 1 with the random numbers to be generated for the simulation.
- (b) Select a random number generator and create the random numbers to be used in the simulation.
- (c) Associate the generated random numbers with the factors identified in step 1 and coded in Step 4 (a).

#### Step 5 Summarize and examine the results obtained in Step 4.

#### Step 6 Evaluate the results of the simulation.

### Generation of Random Numbers

Monte Carlo simulation needs the generation of a sequence of random numbers, which constitute an integral part of the simulation model and also help in determining random observations from the probability distribution.

Random numbers may be found through a computer using random tables or manually. The most common method to obtain random numbers is to generate them through a computer program.

**Example 6.19:** A sample of 100 arrivals of a customer at a retail sales depot is according to the following distribution.

Time between Arrival (min.)	Frequency
0.5	2
1.0	6
1.5	10
2.0	25
2.5	20
3.0	14
3.5	10
4.0	7
4.5	4
5.0	2

A study of the time required to service customers by adding up the bills, receiving payments and placing packages, yields the following distribution.

Time between arrival (min.)	Frequency
0.5	12
1.0	21
1.5	36
2.0	19
2.5	7
3.0	5

Estimate the average percentage of customer waiting time and average percentage of idle time of the server, by simulation, for the next 10 arrivals.

**Solution:** Tag-numbers are allocated to the events in the same proportions as indicated by the probabilities.

Arrivals	Frequency	Probability	Cumulative Probability	Tag-Numbers
0.5	2	0.02	0.02	00-01
1.0	6	0.06	0.08	02-07
1.5	10	0.10	0.18	08-17
2.0	25	0.25	0.43	18-42
2.5	20	0.20	0.63	43-62
3.0	14	0.14	0.77	63-76
3.5	10	0.10	0.87	77-86
4.0	7	0.07	0.94	87-93
4.5	4	0.04	0.98	94-97
5.0	2	0.02	1.00	98-99

Service time (min.)	Frequency	Probability	Cumulative Probability	Tag-Numbers
0.5	12	0.12	0.12	00-11
1.0	21	0.21	0.33	12-32
1.5	36	0.36	0.69	33-68
2.0	19	0.19	0.88	69-87
2.5	7	0.07	0.95	88-94
3.0	5	0.05	1.00	95-99

The random numbers are generated and linked to the appropriate events. The first 10 random numbers, simulating arrival, the second 10, simulating service times. The results are incorporated in Table 6.1, on the assumption that the system starts at 0.00 am.

Average waiting time per customer is,  $\frac{4.5}{10} = 0.45$  minutes.

Average waiting time (or) idle time of the servers =  $\frac{7.00}{10} = 0.7$  minutes.

Table 6.1

Arrival No.	Random Number	Inter-Arrival Time (min.)	Arrival Time (min.)	Random No.	Service Time (Min.)	Service Start	Service End	Waiting Time of End Customer Server	
		<i>a</i>			<i>e</i>	<i>b</i>	<i>f</i>	<i>c = b - a</i>	<i>d = f - a</i>
1	93	4.0	4.0	78	2.0	4	6	-	4.0
2	22	2.0	6.0	76	2.0	6	8	-	-
3	53	2.5	8.5	58	1.5	8.5	10.0	-	0.5
4	64	3.0	11.5	54	1.5	11.5	13	-	1.5
5	39	2.0	13.5	74	2.0	13.5	15.5	-	0.5
6	07	1.0	14.5	92	2.5	15.5	18	1.0	-
7	10	1.5	16.0	38	1.5	18.0	19.5	2.0	-
8	63	3.0	19.0	70	2.0	19.5	21.5	0.5	-
9	76	3.0	22.0	96	3.0	22.0	25.0	-	0.5
10	35	2.0	24.0	92	2.5	25.0	27.5	1.0	-
Total								4.5	7.0

**Example 6.20:** A tourist car operator finds that during the past few months, the car's use has varied so much that the cost of maintaining the car varied considerably. During the past 200 days the demand for the car fluctuated as below.

Trips Per Week	Frequency
0	16
1	24
2	30
3	60
4	40
5	30

Using random numbers, simulate the demand for a 10-week period.

Trips/Week or Demand/Week	Frequency	Probability	Cumulative Probability	Tag-Numbers
0	16	0.08	0.08	00-07
1	24	0.12	0.20	08-19
2	30	0.15	0.35	20-34
3	60	0.30	0.65	35-64
4	40	0.20	0.85	65-84
5	30	0.15	1.00	85-99

**Solution:** The tag-numbers allotted for the various demand levels are shown in the table above.

Week	Random Number	Demand
1	82	4
2	96	5
3	18	1
4	96	5
5	20	2
6	84	4
7	56	3
8	11	1
9	52	3
10	03	0
Total		28

The simulated demand for the cars for the next 10 weeks period is given in the table above.

Total demand = 28 cars.

Average demand =  $\frac{28}{10} = 2.8$  cars per week.

experience indicates the daily demand as given below.

Daily Demand	5	10	15	20	25	30
Probability	0.01	0.20	0.15	0.50	0.12	0.02

Simulate the demand for the next 10 days. Also find the daily average demand for the product on the basis of simulated data.

Solution

Demand	Probability	Cumulative Probability	Tag-Numbers
5	0.01	0.01	00-00
10	0.20	0.21	01-20
15	0.15	0.36	21-35
20	0.50	0.86	36-85
25	0.12	0.98	86-97
30	0.02	1.00	98-99

Day	Random Number	Demand
1	82	20
2	96	25
3	18	10
4	96	25
5	20	10
6	84	20
7	56	20
8	11	10
9	52	20
10	03	10

Total = 170

Average demand =  $\frac{170}{10} = 17$  units/day.

**Example 6.22:** An automobile production line turns out about 100 cars a day but deviations occur owing to many causes. The production is more accurately described by the probability distribution given below.

Production/Day	Probability	Production/Day	Probability
95	0.03	101	0.15
96	0.05	102	0.10
97	0.07	103	0.07
98	0.10	104	0.05
99	0.15	105	0.03
100	0.20	106	0.60
			1.00

Finished cars are transported across the bay at the end of each day by a ferry. If the ferry has space for only 101 cars, what will be the average number of cars waiting to be shipped and what will be the average number of empty space on the ship?

Solution: The tag-numbers are established in the table below.

Production/Day	Probability	Cumulative Probability	Tag-Numbers
95	0.03	0.03	00-02
96	0.05	0.08	03-07
97	0.07	0.15	08-14
98	0.10	0.25	15-24
99	0.15	0.40	25-39
100	0.20	0.60	40-59
101	0.15	0.75	60-74
102	0.10	0.85	75-84
103	0.07	0.92	85-91
104	0.05	0.97	92-96
105	0.03	1.00	97-99

The simulated production of cars for the next 15 days is given in the following table.

Day	Random Number	Production per Day	No. of Cars Waiting	No. of Empty Space in the Ship
1	97	105	4	-
2	02	95	-	6
3	80	102	1	-
4	66	101	-	-
5	96	104	3	-
6	55	100	-	1
7	50	100	-	1
8	29	99	-	2
9	58	100	-	1
10	51	100	-	1
11	04	96	-	5
12	86	103	2	-
13	24	98	-	3
14	39	99	-	2
15	47	100	-	1
Total			10	23

Average number of cars waiting to be shipped =  $\frac{10}{15} = 0.67$  per day.

Average number of empty spaces on the ship =  $\frac{23}{15} = 1.53$  per day.

**Example 6.23:** Suppose that the sales of a particular item per day is Poisson with mean five, then generate 20 days of sales by the Monte Carlo method.

Solution: The probability for the sales is given by,

$$P(X=r) = \frac{e^{-\lambda} \lambda^r}{r!} = \frac{e^{-5} 5^r}{r!} \quad (\because \lambda = 5)$$

NOTES

In Model

N

$\gamma$	Cumulative Probability	Tag-Numbers
0	0.01	00-00
1	0.04	01-03
2	0.13	04-12
3	0.27	13-26
4	0.44	27-43
5	0.62	44-61
6	0.76	62-75
7	0.87	76-86
8	0.93	87-92
9	0.97	93-96
10	0.98	97-97
11	0.99	98-98
12	1.00	99-99

The simulated sales for the next 20 days is given in the table below.

Day	Random Number	Sales	Day	Random Number	Sales
1	49	05	11	99	12
2	55	05	12	89	08
3	89	08	13	10	02
4	15	03	14	27	04
5	12	02	15	50	05
6	94	09	16	93	09
7	85	07	17	92	08
8	34	04	18	57	05
9	07	02	19	50	05
10	53	05	20	78	07

### 6.3.1 Simulation of a Queueing System Using Event List

Simulation is referred as a powerful and extensively used management science technique for the analysis and study of complex systems. Simulation technique is specifically used to imitate the function of a real world system that develops over time. Usually this is performed by developing a simulation model. As a rule, the simulation model typically obtains the form of a set of assumptions with reference to the function of the system and is expressed as mathematical or logical relations between the significant/concerned objects in the system.

Thus, simulation is the process of designing a mathematical or logical model of a real system and then accomplishing experiments with the model to describe, explain and predict the behavior of the real system.

Therefore, to model a system the basic concept of a system must be clear. By definition, a system is a collection of entities that act and interact toward the accomplishment of some logical end. Generally, the systems are referred to be dynamic as their status changes over time. Hence, to explain this status the concept

of the state of a system is used. Simulation is very useful in understanding the expected performance of the real system and also in testing the efficiency of the system design. A static simulation model represents a system at a particular point in time whereas a dynamic simulation model represents a system developed over time.

Based on these two classifications, a simulation may be classified as deterministic or stochastic. A deterministic simulation model is one that contains no random variables whereas a stochastic simulation model contains one or more random variables.

An event is characteristically defined as a situation that causes the state of the system to change instantaneously. When the state of system transforms/changes only at discrete points in time termed as 'discrete events'. When the state of system changes continuously over time, then it is termed as 'continuous event'. All the information regarding change of state is preserved in a list called the event list. In a simulation, time is maintained using a variable termed as the clock time. This simulation is initiated with an empty system and arbitrarily assumed that the first event is an arrival that occurs at clock time 0. Subsequently the departure time of the first customer is scheduled.

$$\text{Departure Time} = \text{Clock Time Now} + \text{Generated Service Time}$$

Next step is to schedule the next arrival into the system by randomly generating an inter-arrival time from the inter-arrival time distribution and setting the arrival time as follows:

$$\text{Arrival Time} = \text{Clock Time Now} + \text{Generated Inter-arrival Time}$$

Both these events have their own specific scheduled times which are maintained on the event list. In a system when events occur, time is advanced from event to event. This approach of simulation is termed as the next-event time-advance mechanism, because of the manner the clock time is updated. The simulation clock can be advanced to the time of the most imminent event.

Moving from event to event, the appropriate actions for each event is carried out including any scheduling of future events. The jump to the next event in the next-event mechanism may be a large one or a small one which specifies that the size of jumps can be changeable or varied. This approach can be distinguished with the fixed-increment time-advance method. With this method, the simulation clock is advanced in increments of  $\Delta t$  time units, where  $\Delta t$  is some appropriate time unit, usually 1 time unit.

However, for most of the models the next-event mechanism can be used to develop models. To demonstrate the computerized simulation model, the following variables must be defined:

$$TM = \text{Clock time of the simulation.}$$

$$AT = \text{Scheduled time of the next arrival.}$$

$$DT = \text{Scheduled time of the next departure.}$$

$$SS = \text{Status of the server (1 = Busy, 0 = Idle).}$$

$WL$  - Length (in time units) of a simulation run.

The simulation can be initiated by initializing all the variables to analyse a specific problem.

**Event Scheduling:** Time is advanced from event to event. The future events list illustrates the ordered list of upcoming events. The moment events are scheduled, they are added to the event list. Similarly, when events occur they are removed from event list.

### 6.3.2 Multiplicative Congruential Algorithm

A Linear Congruential Generator (LCG) represents one of the oldest and well known pseudorandom number generator algorithms. The generator is defined by the recurrence relation:

$$X_{n+1} \equiv (aX_n + c) \pmod{m}$$

Where,

$X_n$  = The sequence of pseudorandom values.

$m, 0 < m$  = The modulus.

$a, 0 < a < m$  = The multiplier.

$c, 0 \leq c < m$  = The increment.

$X_0, 0 \leq X_0 < m$  = The start value or seed.

All these are integer constants that specify the generator. If  $c = 0$ , then the generator is called a *multiplicative congruential method* or Lehmer RNG. If  $c \neq 0$ , then the generator is called a *mixed congruential method*.

If  $c = 0$ , then we have a *power residue or multiplicative generator*.

Note that  $Z_n = (aZ_{n-1}) \pmod{m} \Rightarrow Z_n \neq (a^n Z_0) \pmod{m}$ .

If  $m = 2^B$ , where  $B = \#$  bits in the machine, then the longest period is  $m/4$  if and only if:

- $Z_0$  is odd.
- $a = 8k + 3, k \in \mathbb{Z}^+$  (5, 11, 13, 19, 21, 27, ...)

### 6.3.3 Inverse Transformation Method

In simulation process the inverse transformation method can be described on the basis of following two postulates.

**Inverse Transform Method for Simulating Continuous Random Variables**

Let  $X$  be a random variable with continuous decreasing function  $F_X(x)$ . Since  $F_X(x)$  is a non-decreasing function, the inverse function  $F_X^{-1}(y)$  may be defined for any value of  $y$  between 0 and 1 as follows:

$$F_X^{-1}(y) = \inf\{x: F_X(x) \geq y\} \quad 0 \leq y \leq 1$$

$F_X^{-1}(y)$  is defined to equal that value  $x$  for which  $F(x) = y$ .

**Theorem 1.** If  $U$  is uniformly distributed over the interval  $(0, 1)$ , then  $X = F_X^{-1}(U)$  has continuous decreasing function  $F_X(x)$ .

**Proof:** We have,

$$P(X \leq x) = P(F_X^{-1}(U) \leq x) = P(U \leq F_X(x)) = F_X(x).$$

So to get a value, say  $x$ , of a random variable  $X$ , obtain a value, say  $u$ , of a random variable  $U$ , compute  $F_X^{-1}(U)$ , and set it equal to  $x$ .

**Example 6.24:** Generate a random variable from the uniform distribution  $U(a, b)$ :

$$F_X(x) = \begin{cases} 1/(b-a) & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases}$$

**Solution:** The continuous decreasing function is,

$$F(x) = \begin{cases} 0 & x < a \\ \frac{x-a}{b-a} & a \leq x \leq b \\ 1 & x > b \end{cases}$$

$$U = \frac{x-a}{b-a}$$

$$X = F^{-1}(U) = a + (b-a)U$$

### Inverse Transform Method for Simulating Discrete Random Variables

The inverse transform method for simulating from continuous random variables have analog in the discrete case. For example, to simulate a random variable  $X$  having partial differential function,

$$P(X = x_j) = P_j, \quad j = 0, 1, \dots$$

$$\sum_j P_j = 1$$

$$F(X_j) = \sum_{i=0}^j P_i$$

To simulate  $X$  for which  $P(X = x_j) = P_j$ , consider that  $U$  be uniformly distributed over  $(0, 1)$  and set as,

$$X = \begin{cases} x_0 & U < P_0 \\ x_1 & P_0 \leq U < P_0 + P_1 \\ \dots & \dots \\ x_j & \sum_{i=0}^{j-1} P_i \leq U < \sum_{i=0}^j P_i \end{cases}$$

NOTES

Since,

$$P(X = x_j) = P\left(\sum_{i=0}^{x_j-1} P_i \leq U \leq \sum_{i=0}^{x_j} P_i\right) = \int_{F(x_{j-1})}^{F(x_j)} dx = F(x_j) - F(x_{j-1}) = P_j$$

In this case  $X$  has the desired distribution.

**Example 6.25:** Simulating a Poisson random variable.

**Solution:** The random variable  $X$  is Poisson with mean  $\lambda$  if,

$$p_i = P(X = i) = e^{-\lambda} \frac{\lambda^i}{i!} \quad \text{where, } i = 0, 1, \dots$$

The key to use the inverse transform method is to generate such a random variable with the following identity:

$$P_{i+1} = \frac{\lambda}{i+1} P_i, \quad \text{where, } i \geq 0$$

Using this recursion to compute the Poisson probabilities, the inverse transform algorithm for generating a Poisson random variable with the mean  $\lambda$  can be expressed as follows.

The quantity  $i$  refers to the value presently under consideration;  $p = p_i$  is the probability that  $X$  is equal to  $i$ , and  $F = F(i)$  is the probability that  $X$  is less than or equal to  $(i)$ . Now follow the given steps:

*Step 1:* Generate a random number  $U$ .

*Step 2:* Let  $i = 0$ ,  $p = e^{-\lambda}$ ,  $F = p$ .

*Step 3:* If  $U < F$ , set  $X = i$  and stop.

*Step 4:* Let  $p = \lambda p / (i + 1)$ ,  $F = F + p$ ,  $i = i + 1$ .

*Step 5:* Go to Step 3.

This algorithm generates a Poisson random variable with the mean  $\lambda$ . Note that it first generates a random number  $U$  and then checks whether or not  $U < e^{-\lambda} = p_0$ . If so, it sets  $X = 0$ . If not, then it computes  $p_i$  by using the recursion method. It now checks whether  $U < p_0 + p_1$  and if so it sets  $X = 1$ , and so on.

Thus, the algorithm successively checks whether the Poisson value is 0, then whether it is 1, then 2, and so on. The number of comparisons required will be 1 greater than the generated value of the Poisson. Hence, the above algorithm will have  $1 + \lambda$  searches. It now generates a Poisson random variable with the mean  $\lambda$  by generating a random number  $U$ .

## 6.4 MONTE-CARLO SIMULATION

Monte Carlo simulation is often used by modern management when it cannot use other techniques. There are many industrial problems which defy mathematical solutions. The reason is that either they are too complicated or that the data cannot be

expressed in mathematical terms. In such cases, it is still possible to reach valid conclusions by using the Monte Carlo technique. A considerable help is thus obtained at practically no cost in taking decisions concerning the functioning of a business system. The data and conclusions can be obtained through simulation of an actual operation on the basis of its own past working. It paves the way for predicting the changes in its behaviour and the result is evaluated from innovations that we want to introduce.

By using a fresh series of random numbers at the appropriate junctures we can also examine the reactions of the simulated model just as if the same alterations had actually been made in the system itself. Monte Carlo simulation, therefore, provides a tool of knowing in advance whether or not the expense to be incurred or the investment to be made in making the changes envisaged. Through this technique, you can introduce the innovations on a piece of paper, examine their effects and then may decide to adopt or not to adopt such innovations in the functioning of real system. The usefulness of simulation lies in the fact that it allows us to experiment with a model of the system rather than the actual system; in case we are convinced about the results of our experiments we can put the same into practice. Thus the effect of the actual decisions are tested in advance through the technique of simulation by resorting to the study of the model representing the real life situation or the system.

The main purpose of simulation in management is to provide feedback, which is vital for the learning process. It creates an atmosphere in which managers play a dynamic role by enriching their experience through involvement in reckoning with actual conditions through experimentation on paper. The technique permits trying out several alternatives as the entire production for service process can be worked out on paper, without dislocating the system in any way. Thus, Monte Carlo technique transforms the manager from a blind-folded driver of an automobile, reacting to instructions of a fellow passenger to one who can see fairly, clearly, where he is going.

### Introduction to Simulation

Simulation is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions.

Simulation is also defined as the use of a system model that has the designed characteristics of reality in order to produce the essence of actual operation.

Table 6.2 is an example of simulation worksheet.

Table 6.2 Simulation Worksheet for Simulating Sizes to Locate the Number of Misfits

Assembly S.N.	Shafts Random Numbers (from Tippett Tables)	Random Normal Deviate (z)	Simulating Size $X = \mu + z(\sigma)$	Rings Random Numbers (from Tippett Tables)	Random Normal Deviate (z)	Simulating Size $X = \mu + z(\sigma)$
1	2952	0.82	$0.980 + 0.82(0.01) = 0.9882$	3992	1.28	$1.0 + 1.28(0.02) = 1.0256$
2	3170	0.91	$0.980 + 0.91(0.01) = 0.9891$	4167	1.38	$1.0 + 1.38(0.02) = 1.0276$

3	7203	$-0.59 \ 0.980 - 0.59(0.01)$	1300	0.33	$1.0 + 0.33(0.02)$
		$= 0.9741$			$= 1.0066$
4	3408	$1.00 \ 0.980 + 1.0(0.01)$	3563	1.06	$1.0 + 1.06(0.02)$
		$= 0.9900$			$= 1.0212$
5	6560	$0.14 \ 0.980 + 0.14(0.01)$	1112	0.28	$1.0 + 0.28(0.02)$
		$= 0.9814$			$= 1.0056$
6	6641	$-0.42 \ 0.980 - 0.42(0.01)$	9792	-2.04	$1.0 - 2.04(0.02)$
		$= 0.9758$			$= 0.9592$
7	5624	$-0.16 \ 0.980 - 0.16(0.01)$	9525	-1.67	$1.0 - 1.67(0.02)$
		$= 0.9784$			$= 0.9666$
8	5356	$-0.09 \ 0.980 - 0.09(0.01)$	2693	0.74	$1.0 + 0.74(0.02)$
		$= 0.9791$			$= 1.0148$
9	2769	$0.76 \ 0.980 + 0.76(0.01)$	6107	-0.28	$1.0 - 28(0.02)$
		$= 0.9876$			$= 0.9944$
10	5246	$-0.06 \ 0.980 - 0.06(0.01)$	9025	-1.29	$1.0 - 1.29(0.02)$
		$= 0.9794$			$= 0.9742$

### Areas of Application of Monte Carlo Simulation

Monte Carlo simulation has been applied to a wide diversity of problems ranging from queuing process, inventory problem, risk analysis concerning a major capital investment such as the introduction of a new product, expansion of the capacity, and many other problems. Budgeting is another area where simulation can be very useful. In fact, the system of flexible budgeting is an exercise in simulation. Simulation can as well be used for preparing the master budget through functional budgets.

Over and above, the greatest contribution of simulation is in the analysis of complex systems. Many real-world problems involve systems made up of many components parts that are interrelated. The system may be dynamic and changing over time and may involve probabilistic or uncertain events. Simulation is the only technique for quantitative analysis of such problems.

### Monte Carlo Method

Monte Carlo methods are basically the algorithms used in the computation of result to be calculated from repeated random sampling. These methods help in computerized calculations because these can perform repeated computation using random or pseudo-random numbers. It is also used when it is not feasible to compute correct result with a deterministic algorithm. Monte Carlo simulation methods are used to study systems having degrees of freedom and in the situations when there is significant ambiguity in inputs for example, calculating risk factor in a business.

Various simulation models, based on the principle of similitude (such as model of aeroplanes initiating flight conditions in a wind tunnel) have been in use for a long time. However, Monte Carlo simulation is a recent operations research innovation. The novelty lies in making use of pure chance to contact a simulated

version of the process under analysis, in exactly the same way as pure chance operates the original system under working conditions. Only models under uncertainty can be evaluated using Monte Carlo technique.

'Monte Carlo' is the code name given by John von Neumann and S.M. Ulam to the technique of solving problems though it is too expensive for experimental solutions and too complicated for analytical treatment.

Monte Carlo method is not one single method. It involves various widely-used classes of approaches to follow a specific model. Using it the following can be done:

- Define a domain with feasible inputs.
- Randomly generate inputs from the domain.
- Perform deterministic computation with the inputs.
- Combine the results of the personal computations into the final result.

Monte Carlo methods are used to solve various mathematical problems based on sampling experiments in statistics using the sequences of random numbers for simulation and are termed as statistical simulation methods. Thus, Monte Carlo method is not one single method, but it is a collection of various methods and is basically used to perform similar procedure. Some of these methods are discussed here with the help of solved examples.

The Monte Carlo simulation technique can as well be used to solve probabilistic problems. Suppose, we are to evaluate the probability  $P$  that a tank will be knocked out by either a first or second shot from an antitank gun assumed to possess a constant kill probability of  $1/2$ . The probability analysis will say that the chance of tank being knocked out by either a first or second shot from an antitank gun is  $1/2 + 1/2(1 - 1/2) = 3/4$ . However, we can also work out this probability by simulating each round of the antitank gun by the flip of a coin through Monte Carlo simulation technique.

Since the probability of a 'head' is the same as that of a kill, we may call it a hit when the coin turns up a head and otherwise a miss. If we flip the coin a large number of times, the value of  $P$  may be calculated by merely counting the number of times a head turns up at least in two successive throws and then dividing this number by the total pairs of throws of the coin. Monte Carlo method in this simple case is indeed a poor substitute for the theoretical probability analysis. However, many real-life systems are so complicated that even the well defined probability analysis very often fails but such situations can be handled by Monte Carlo simulation, particularly the Monte Carlo technique that provides the simplest possible solutions for queuing problems. Problems of corporate planning, inventory control, capital investment, consumer behaviour and quality control can also be handled through simulation.

Monte Carlo simulation uses random number tables to reproduce on paper the operation of any given system under its own working conditions. This technique is used to solve problems that depend upon probability where formulation of mathematical model is not possible. It involves first, the determining of the

NOTES

probability distribution of the concerned variables and then sampling from this distribution by means of random numbers to obtain data. It may, however, be emphasized here that the probability distributions to be used should closely resemble the real world situation.

One should always remember that simulation is not a perfect substitute but rather an alternative procedure for evaluating a model. Analytical solution produces the optimal answer to a given problem, while Monte Carlo simulation yields a solution which should be very close to the optimal but not necessarily the exact correct solution. Monte Carlo Simulation solution converges to the optimal solution as the number of simulated trials goes to infinity.

**Importance and Uses of Simulation Technique**

*Monte Carlo Simulation in the Context of Queuing Theory*

Monte Carlo simulation which uses random number tables can better be illustrated by considering any concrete operation subject to chance. Let us take the arrival of scooters at a service station. First of all, observe the actual arrivals of scooters on number of days, say for five days, then put this information in the following two ways:

- Group the number of scooters arriving every hour, say between 7-8, 8-9 a.m. and so on till 4-5 p.m. (assuming the working hours of the service station is from 7 a.m. to 5 p.m. and also assuming that no scooter arrives before 7 a.m. nor any scooter after 5 p.m.). Work out the mean number of scooters arriving during 7-8 a.m., 8-9 a.m., 9-10 a.m. and so on. Let us suppose, we get the information as illustrated in Table 6.3.

*Table 6.3 Mean Number of Scooters Arriving Per Hour*

7-8 a.m.	5.6
8-9 a.m.	5.4
9-10 a.m.	3.4
10-11 a.m.	3.6
11-12 noon	2.0
12-1 p.m.	3.0
1-2 p.m.	4.0
2-3 p.m.	6.0
3-4 p.m.	3.0
4-5 p.m.	4.0

- Obtain the deviation of actual arrivals during a particular hour from the corresponding mean and do it for all the hours from 7 a.m. to 5 p.m. There will thus be 10 sets of 5 deviations (because of the observation on 5 days) from each of the 10 mean hourly arrivals and then prepare a frequency distribution of such deviations. Suppose we get the frequency distribution of such deviations as illustrated in Table 6.4.

*Table 6.4 Random Number Allotment*

<i>Deviation from Mean</i>	<i>Frequency</i>	<i>Percentage Frequency</i>	<i>Probability</i>	<i>Random Nos. Allotted</i>
+4.0	2	4	0.04	00-03
+3.5	3	6	0.06	04-09
+3.0	5	10	0.10	10-19
+2.5	4	8	0.08	20-27
+2.0	6	12	0.12	28-39
+1.5	2	4	0.04	40-43
+1.0	3	6	0.06	44-49
+0.5	4	8	0.08	50-57
-0.5	3	6	0.06	58-63
-1.0	2	4	0.04	64-67
-1.5	4	8	0.08	68-75
-2.0	2	4	0.04	76-79
-2.5	3	6	0.06	80-85
-3.0	1	2	0.02	86-87
-3.5	2	4	0.04	88-91
-4.0	4	8	0.08	92-99
Total	50	100	1.00	

In the last column of the Table 6.4 we have allotted 100 random numbers from 00 to 99, both inclusive according to the percentage frequency of the deviation or the probability distribution of deviations. Thus deviation to the extent of +4.0 from the mean, having a frequency of 2 out of 50 (and as such 4 out of 100) or a probability of 0.04, has been allotted 4 random numbers, 00 to 03. The next deviation of +3.5 with a probability of 0.06 has the next 6 random numbers, 04 to 09, allotted to it. The same treatment has been done to all the remaining deviations. The last deviation - 4.0 with a probability of 0.08 has been allotted to the last 8 random numbers, from 92 to 99.

The object of doing all this is to derive by simulation the actual number of scooters that may be expected to arrive during any given hour. Suppose we want to know the expected number of scooters arriving on a particular day during the hour 8 to 9 a.m. The table giving the mean number of scooters arrival shows that the mean arrival during this hour is 5.4. If we can ascertain the deviation of the actual arrivals from the mean we can easily work out the actual number of scooters arrived. To do so, we look at the table of random numbers and select any two-digit number at random. Suppose, the random number so selected is 84 corresponding to which the value of the deviation of actual arrivals from the mean as per the above table is -2.5. In other words the actual arrivals for the hour 8-9 a.m. will be  $(5.4) - (2.5) = 2.9$  (or approximately 3), so that we may say that three scooters will arrive between 8-9 a.m. on the day in question. The underlying rationale of this simulation procedure is that every deviation from its corresponding mean has the same chance of occurring by random number selection as in the actual case because each deviation has as many random numbers allotted to it as its frequency percentage in the general pool of all deviations as stated above.

simulate and tell the actual arrival time of the scooter coming to the service station during any particular hour. If we are satisfied to note the arrival time correct to within, say, 5 minutes the required numbers of minutes past the hour can take a value only in one of the 12 intervals (0-5, 6-10, 11-15, ..., 56 to 60 minutes) into which any hour can be divided. Keeping this in view and the actual observations for all five days under consideration we can prepare a frequency table showing how many scooters arrive within 5 minutes, how many within 6-10 minutes past the hour and so on for the remaining intervals into which we choose to split the hour. Let the observed information for 5 days period on this basis be put as shown in Table 6.5.

Random numbers in the last column of the Table 6.5 have been allotted in a similar manner as we did in an earlier table. We have already seen that three scooters arrive during the hour 8-9 a.m. and now we want to know the actual time of their coming to service station.

**Table 6.5** Number of Scooters Arriving Within the Number of Minutes Past the Hour

Deviation from Mean	Frequency	Percentage Frequency	Probability	Random Nos. Allotted
0-5 mts.	20	10	0.10	00-09
6-10 mts.	30	15	0.15	10-24
11-15 mts.	10	5	0.05	25-29
16-20 mts.	40	20	0.20	30-49
21-25 mts.	16	8	0.08	50-57
26-30 mts.	14	7	0.07	58-64
31-35 mts.	18	9	0.09	65-73
36-40 mts.	12	6	0.06	74-79
41-45 mts.	16	8	0.08	80-87
46-50 mts.	14	7	0.07	88-94
51-55 mts.	6	3	0.03	95-97
56-60 mts.	4	2	0.02	98-99
<b>Total</b>	<b>200</b>	<b>100</b>	<b>1.00</b>	

For this purpose we pick a two-digit random number from the table of random numbers and let us say it is 25. A reference to the above table shows that this number occurs in the range of 25-29 which belongs to the interval 11-15 minutes and this means that the first of the 3 scooters arriving between 8-9 a.m. arrives at 15 minutes past 8 a.m. Similarly picking two more random numbers, viz., 36 and 96 we find from the above table that they are related to intervals 16-20 minutes and 51-55 minutes respectively. Thus the second scooter arrives at 20 minutes past 8 a.m. and the third scooter arrives at 55 minutes past 8 a.m.

Proceeding in a similar manner we can also make a frequency table showing the number of scooters serviced within intervals of varying magnitudes. Each of these intervals can then be allotted a set of 100 random numbers (i.e., 00 to 99) according to the probability distribution to provide a basis for simulating the pattern of available service.

The mentioned is an example of how arrival as well as service pattern in a queuing process may be derived by Monte Carlo simulation. Remember that there is no regularity either in the arrival of the scooters or in rendering service and because of this there may be times when scooters have to wait for service while at other times the service attendant may remain idle. If in such a case we want to add one more service point to the service station then certainly we would first like to assess whether the same would be economical or not. Simulation technique can assist us in the matter. How does simulation help can better be made clear by the following example:

**Example 6.26:** A firm has a single channel service station with following empirical data available to its management:

- (i) The mean arrival rate is 6.2 minutes
- (ii) The mean service time is 5.5 minutes
- (iii) The arrival and service time probability distributions are as follows:

Arrivals (Minutes)	Probability	Service Time (Minutes)	Probability
3-4	0.05	3-4	0.10
4-5	0.20	4-5	0.20
5-6	0.35	5-6	0.40
6-7	0.25	6-7	0.20
7-8	0.10	7-8	0.10
8-9	0.05	8-9	0.00
	<b>1.00</b>		<b>1.00</b>

The queuing process begins at 10.00 a.m. and proceeds for nearly 2 hours. An arrival goes to the service facility immediately if it is empty otherwise it will wait in a queue. The queue discipline is, First come first served.

If the attendant's wage is ₹ 8 per hour and the customer's waiting time cost ₹ 9 per hour, would it be an economical proposition to engage second attendant? Answer on the basis of Monte Carlo simulation technique. You may use the figures based upon the simulated period for 2 hours.

**Solution:** From the given probability distributions of arrivals and service times, first of all we allot the random numbers to the various intervals. This has been done as follows:

Arrivals (Minutes)	Probability	Random Nos. Allotted	Service Time (Minutes)	Probability	Random Nos. Allotted
3-4	0.05	00-04	3-4	0.10	00-09
4-5	0.20	05-24	4.5	0.20	10-29
5-6	0.35	25-59	5-6	0.40	30-69
6-7	0.25	60-84	6-7	0.20	70-89
7-8	0.10	85-94	7-8	0.10	90-99
8-9	0.05	95-99	8-9	0.00	

These allotted random numbers now become the basis for generating arrival and service times in conjunction with a table of random numbers. A simulation worksheet as shown on the next page has been developed in the following manner:

As the given mean arrival rate is 6.2 minutes which means that in an hour approximately 10 units arrive and as such in a simulation period for 2 hours about 20 units are expected to arrive. Hence, the number of arrivals for our simulation exercise is 20.

A table of random numbers (given in Table 6.5) is used for developing the simulation worksheet. The first random number for arrival time is 44. This number lies between 25-59 and this indicates a simulated arrival time of 5 minutes. All simulated arrival and service times have been worked out in a similar fashion.

The next step is to list the arrival time in the appropriate column. The first arrival comes in 5 minutes after the starting time. It means the attendant waited for 5 minutes. This has been shown under the column 'Waiting Time: Attendant'. The simulated service time for the first arrival is 5 minutes which results in the service being completed by 10.10 a.m. The next arrival comes at 10.11 a.m. which indicates that no one has waited in queue but the attendant has waited for 1 minute from 10.10 a.m. to 10.11 a.m. The service time ends at 10.18 a.m. However, the third arrival comes at 10.17 a.m. and the service of the second continues upto 10.18 a.m., hence the third arrival has to wait in the queue. This is shown in the column 'Waiting Time: Customer' of the simulation worksheet. One customer waiting in queue is shown in the column—'Length of the Line.' The same procedure has been followed throughout the preparation of the simulation worksheet.

The following information can be derived from the above stated simulation worksheet.

1. Average length of queue:

$$= \frac{\text{No. of customers in line}}{\text{No. of arrivals}} = \frac{13}{20} = 0.65$$

2. Average waiting time of customer before service:

$$\frac{\text{Customer waiting time}}{\text{No. of arrivals}} = \frac{41}{20} = 2.05 \text{ minutes}$$

3. Average service time:

$$= \frac{\text{Total service time}}{\text{No. of arrivals}} = \frac{107}{20} = 5.35 \text{ minutes}$$

4. Time a customer spends in the system:

$$= \text{Average service time} + \text{Average waiting time before service}$$

$$= 5.35 + 2.05 = 7.40 \text{ minutes.}$$

Simulation worksheet developed above also states that if one more attendant is added then there is no need for a customer to 'Wait in Queue'. But the cost of having one more attendant in addition to the existing one is to be compared with the cost of one attendant and the customer waiting time. This can be worked out as under:

Two Hour Period	Cost with	
	One Attendant	Two Attendants
Customer waiting time (41 mts. × ₹ 9 per hour)	₹ 6.15	nil
Attendant's cost (2 hours × ₹ 8 per hour)	₹ 16	₹ 32
<b>Total</b>	<b>₹ 22.15</b>	<b>₹ 32</b>

If the above analysis is based on simulation for a period of 2 hours and is representative of the actual situation, then it can be concluded that the cost with one attendant is lower than with two attendants. Hence, it would not be an economical proposition to engage additional attendant.

### Monte Carlo Simulation in the Context of Inventory Control

The following example explains the concept of Monte Carlo simulation in the content of inventory control.

**Example 6.25:** Suppose that the weekly demand of Electric Motors has the following probability distribution:

Number Demanded	Probability	Random Numbers Assigned
0	0.10	00 to 09
1	0.40	10 to 49
2	0.30	50 to 79
3	0.20	80 to 99
<b>Total</b>	<b>1.00</b>	

Distribution pattern of delivery time was as follows:

Number of Weeks from Order to Delivery	Probability	Random Numbers Assigned
2	0.20	00 to 19
3	0.60	20 to 79
4	0.20	80 to 99
<b>Total</b>	<b>1.00</b>	

Inventory carrying cost is ₹ 5 per unit per week, order placing cost is ₹ 10 per occurrence and loss in net revenue (sale price less cost of goods) is ₹ 50 per unit from shortage.

Estimate average weekly cost of the inventory system with a policy of using reorder quantities of 4 and a reorder points of 5 units using the technique of Monte Carlo simulation for 20 weeks period taking 8 units as the opening balance of inventory.

**Solution:** We shall first develop the simulation worksheet, keeping all the given information in view, shown follows:

Simulation worksheet for a period of 20 weeks concerning the inventory system and the related costs with reorder point of 5 units and reorder quantity of 4 units.

Simulated Week Number	Random Numbers (using Random Number Tables)		Simulated Weekly			Simulated Costs Activity Shortage			
	For Demand	For Delivery Time	Demand Units	Receipt Units	Balance Units	Inventory Holding (in ₹)	Ordering (in ₹)	Shortage (in ₹)	
1	2	3	4	5	6	7	8	9	
0					8*				
1	44		1		7	35			
2	84	82	3		4	20	10		
3	50		2		2	10			
4	85		3		(-1)			50	
5	40		1		(-1)			50	
6	96	88	3	4	1	5	10		
7	16		1						
8	16		1		(-1)			50	
9	97		3		(-1)			150	
10	92	39	3	4	1	5	10		
11	33		1						
12	83		3		(-3)			150	
13	42	16	1	4	3	15	10		
14	07		0		3	15			
15	77	66	2	4	5	25	10		
16	50		2		3	15			
17	20		1		2	10			
18	50	95	2	4	4	20	10		
19	58		2		2	10			
20	44		1		1	5			
Total			36	20	39	190	60	450	
Average			1.8	1	1.95	9.5	3	22.5	

The procedure adopted in developing the above work sheet is briefly explained as under for the guidance of the readers.

Successive Week Numbers	Occurrence of Events and Their Effects
1	Random number '44' indicates in column of given demand pattern that customers demanded 1 unit leaving an inventory balance of 8 - 1 = 7 units at a carrying cost of ₹ 5 per unit which equals ₹ 35.
2	Random number '84' indicates demand for 3 units and leaves an inventory balance of 4 units which is less than the reorder point of 5 units. It costs ₹ 10 to reorder. New order was placed. Random number '82' indicates that it will take 4 weeks to receive the new units ordered.

3	No special event.
4	There was demand for 3 units but only 2 units were sold because of inventory exhaustion. Under the assumption that customers went elsewhere, one unit not sold represents a loss in net revenue to the extent of ₹ 50.
5	New stock still has not arrived and a further loss of potential sales of 1 unit at a cost of ₹ 50 occurred.
6	The quantity of 4 units ordered in week, two has arrived of which 3 units were sold and 1 unit was in balance. Hence, new order was placed.
7	Neither there was any shortage nor there was anything left in inventory balance.
8 and 9	Shortage to the extent of 1 unit and unit 3 respectively remained causing a potential loss of ₹ 50 and ₹ 150 respectively.
10	Ordered quantity in week six arrived, three units were sold and one remained in balance. Fresh order was placed.
11	No shortage and no balance.
12	Shortage and potential loss of ₹ 150.
13	Ordered quantity in week 10 arrived, 1 unit was sold and three remained in balance. Fresh order was placed.
14	No special event.
15	Ordered quantity in week 13 arrived, 2 units sold and 5 units remained in balance. Fresh order was placed.
16 and 17	No special event.
18	Ordered quantity in week 25 arrived, 2 units sold and 4 remained in balance. Fresh order was placed.
19 and 20	No special event.

Under the assumptions of the simulation procedure it is noted that weekly costs averaged ₹ 9.5 to carry inventory, ₹ 3 to order and ₹ 22.5 from shortages. In all it costs ₹ 35 per week approximately to maintain the inventory system with a policy of using reorder quantities of 4 and a reorder of 5 units.

#### Monte Carlo Simulation in the Context of Production Line Model

**Example 6.28:** A certain production process produces on the average seven per cent defective items. Defective items occur randomly. Items are packaged for sale in lots of five. The production manager wants to know what percentage of the lots contains no defective items. You are required to solve the manager's problem using Monte Carlo simulation. You may as well compare your simulated results with those obtainable using the analytical methods. If there remains any difference between such results then account for the same. (Simulate at least 10 lots each of five items.)

**Solution:** Since on an average the process produces 7 per cent defective items, so 93 per cent of the items produced are good. For this we can have the following probability distribution:

NOTES

Items	Probability	Random Nos. Assigned
Defective	0.07	00 to 06
Good	0.93	07 to 99
Total	1.00	

Using the random number table develop the Simulation Worksheet as follows:

Simulation worksheet for 16 lots taken for finding the percentage of lots with no defective items.

Lots	Random Numbers (Five for Each Lot)					Simulated Items G = Good; D = Defective					Defective Number in Lot
1	44	84	82	50	85	G	G	G	G	G	0
2	40	96	88	16	16	G	G	G	G	G	0
3	97	92	39	33	83	G	G	G	G	G	0
4	42	16	07	77	66	G	G	G	G	G	0
5	50	20	50	95	58	G	G	G	G	G	0
6	44	77	11	08	38	G	G	G	G	G	0
7	87	45	09	99	81	G	G	G	G	G	0
8	97	30	36	75	72	G	G	G	G	G	0
9	79	83	07	00	42	G	G	G	D	G	1
10	13	97	16	83	11	G	G	G	G	G	0
11	45	65	34	89	12	G	G	G	G	G	0
12	64	86	46	32	76	G	G	G	G	G	0
13	07	51	25	36	19	G	G	G	G	G	0
14	32	14	31	96	03	G	G	G	G	D	1
15	93	16	62	24	08	G	G	G	G	G	0
16	38	88	74	47	00	G	G	G	G	D	1

On the basis of these simulated sample of 16 lots we can estimate the percentage

$$\text{of good lots as } = \frac{13}{16} \times 100 = 81.25$$

If we use the analytical method or well formulated mathematical method, we get the following result:

$$p = p. (\text{Good item}) = 0.93$$

$$q = p. (\text{Defective item}) = 0.07$$

Each lot is of 5 items which means that  $n = 5$ .

Using the binomial probability function we have the following:

$$\begin{aligned}
 p \text{ (all five good items)} &= {}^5C_5 (0.93)^5 (0.07)^0 \\
 &= 1(0.93)^5 \cdot 1 \\
 &= 0.70 \\
 &= 70\%
 \end{aligned}$$

This analytical answer equals the true expected value or the average number of good lots over the long run. However, the simulated result gives this percentage as 81.25 which differs from that of 70 per cent obtained by mathematical method. The difference is on account of the fact that we took only 16 lots for simulation exercise. If we increase this number, then our answer will approximate to that of the answer from analytical method. Thus, simulation gives only the best possible estimates and not the optimal result as given by analytical methods.

## 6.5 INVENTORY PROBLEMS

Inventory is defined as any idle resources of an enterprise. It is a physical stock of goods kept for future use. In a factory, the inventory may be in the form of raw materials, parts, semi-finished goods. Inventory also includes furniture, machinery, etc.

### Reasons for Maintaining Inventories

The need of the management to make decisions regarding the inventory arises because of the various alternative courses of action available with the enterprise. It is essential for an enterprise to have inventory for the following reasons.

- (i) It helps in smooth and efficient running of the business.
- (ii) It provides adequate service to the customers.
- (iii) It reduces the possibility of duplication of orders.
- (iv) It helps in maintaining economy by absorbing some of the fluctuations when the demand of an item fluctuates or is seasonal.
- (v) It helps in minimizing the loss due to the deterioration, obsolescence, damage, etc.
- (vi) It acts as a buffer stock when raw materials are received late and shop rejections are too many.
- (vii) It takes advantage of price discounts by bulk purchasing.

Though the inventories are essential and provide an alternative to production/purchase in the future, it also locks up the capital of the enterprise. It includes the expenses of stores, equipment, personnel, insurance, etc., therefore, excess inventories are undesirable. Larger inventories do not necessarily lead to a high volume of output instead it might hamper the production.

Our problem is to balance the advantages of having inventories and the cost of carrying them to arrive at an optimal level of inventories to minimize the total inventory cost. This calls for controlling the inventories in the most profitable way. The basic objective of inventory control is to release capital for more productive use.

### Types of Inventories

There are five types of inventories, which are as follows:

- (i) Transportation inventories
- (ii) Buffer inventories
- (iii) Anticipation inventories
- (iv) Decoupling inventories
- (v) Lot-size inventories

- (i) **Transportation Inventories:** These arise due to the transportation of inventory items to the various distribution centres and customers from the various production centres. The amount of transportation inventory depends on the time consumed in transportation and the nature.
- (ii) **Buffer Inventories:** These are maintained to meet the uncertainty of demand and supply.

Check Your Progress

21. What is the Linear Congruential Generator (LCG) represent?
22. What is the purpose of simulation in management?
23. What are Monte Carlo methods?
24. Write one difference between analytical solution and Monte Carlo simulation.

(iii) **Anticipation Inventories:** These are built in advance by anticipating or foreseeing the future demand, e.g., production of crackers before the diwali festival, electric fans, or coolers before the onset of summer season.

(iv) **Decoupling Inventories:** The inventories used to reduce the interdependence of various stages of production system are known as decoupling inventories.

(v) **Lot-Size Inventories:** Generally, the rate of consumption is different from the rate of production or purchasing. Therefore, items are produced in larger quantities which result in lot-size, also called as cycle inventories.

### Inventory Costs

There are four categories of inventory cost associated with keeping inventories of items. They are as follows:

- (i) Item (or production or purchase) cost
- (ii) Ordering or set-up cost
- (iii) Carrying or holding cost
- (iv) Shortage or stock out cost

(i) **Item Cost:** It refers to the cost associated with an item whether it is manufactured or purchased. The purchase price will be considered when discounts are allowed for any purchase above a certain quantity.

(ii) **Set-Up Cost ( $C_2$ ):** These costs include the fixed cost associated with obtaining the goods through placing of an order or purchasing of manufacturing or setting-up a machinery before starting the production. They include the costs of purchase, requisition, follow up, receiving the goods, quality control, etc. These are also called order costs or replenishment costs and are usually denoted by  $C_2$  per production run (cycle). They are assumed to be independent of the quantity ordered or produced.

(iii) **Carrying or Holding Cost ( $C_1$ ):** The cost associated with carrying or holding the goods in stock is known as holding or carrying cost, which is denoted by  $C_1$  per unit of goods for a unit of time. Holding cost is assumed to vary directly with the size of inventory as well as the time the item is held in stock. The following components constitute the holding cost:

- (a) Invested capital cost which is the interest charge over the capital invested.
- (b) Record keeping and administrative cost.
- (c) Handling cost includes costs associated with the movement of stock, such as cost of labour.
- (d) Storage costs.
- (e) Depreciation costs.
- (f) Taxes and insurance, etc.

If  $P$  is the purchase price of an item,  $I$  is the stock holding cost per unit time as a fraction of stock value then the holding cost is  $C_1 = IP$ .

(iv) **Shortage Cost or Stock-Out Cost ( $C_3$ ):** The penalty costs that are incurred as a result of running out of stock (i.e., shortage) are known as shortage or stock-out costs. These are denoted by  $C_3$  per unit of goods for a specified period.

If the unfilled demand for the goods can be satisfied at a latter date (backlog case), these costs are assumed to vary directly with both the shortage quantity and the delaying time. If the unfilled demand is lost (no back-log case), shortage cost becomes proportional to shortage quantity only.

### Variables in the Inventory Problem

The variables involved in the inventory model are of two types, viz. (i) Controlled variables (ii) Uncontrolled variables.

(i) **Controlled variables:** These variables include the following:

- (a) The quantity of an item that should be ordered
- (b) The frequency or timing of acquisition
- (c) The completion stage of stocked items

(ii) **Uncontrolled variables:** These include holding costs, shortage costs and set-up costs.

**Note:** Total inventory cost = Purchase cost of inventory items + Ordering cost + Carrying cost + Shortage costs.

### Other Factors Involved in Inventory Analysis

#### Demand

Demand refers to the number of items required per period. It may be known exactly or known in terms of probabilities or may be completely unknown.

The demand pattern of items may be either deterministic or probabilistic. Problems in which demand is known and fixed are called deterministic problems. Whereas those problems in which the demand is assumed to be a random variable are called stochastic or probabilistic problems.

In case of deterministic demand, it is assumed that the quantities needed over subsequent periods of time are known exactly. Further, the known demand may be fixed or variable with time. Such demands called state or dynamic demands respectively.

The probabilistic demand occurs when the demand over a certain period of time is not known with certainty, but it is described by a known probability distribution. A probabilistic demand may be either stationary or non-stationary over time.

#### Lead Time

The time gap between the placing of an order and the actual arrival of the inventory is known as lead time. If the lead time is known and is not equal to zero, and if the demand is deterministic, all that one requires to do is to order in advance by the time equal to the lead time. If the lead time is zero, there is no need to order in advance.

In case, the lead time is a variable which is known only probabilistically, then the question of when to order is more difficult. The amount and the timing of replenishment is found by considering the expected costs of holding and shortage over the lead time required.

### Amount Delivered (Supply of Goods)

The supply of goods may be instantaneous or spread over a period of time. If a quantity  $q$  is ordered or purchased or produced, the amount delivered may vary around  $q$  with a known probability density function.

#### Order Cycle

The time period between placement of two successive orders is referred to as an order cycle. The order cycle may be placed on the basis of the following two types of inventory review systems.

- (i) **Continuous Review:** The record of the inventory level is checked continuously until a certain lower limit (known as recorder level) is reached when a new order is placed. This is often known as two-bin system.
- (ii) **Periodic Review:** In this, the inventory levels are reviewed at equal time intervals and orders are placed at such intervals. The quantity ordered each time depends on the available inventory level at the time of review.

#### Time Horizon

The time period over which the inventory level will be controlled is known as time horizon.

#### Recorder Level

The level between the maximum and the minimum stock at which the purchasing (manufacturing) activities must start for the replenishment is known as recorder level.

The inventory model can be classified into two following categories:

- (i) Deterministic inventory model
- (ii) Probabilistic inventory model

#### Economical Order Quantity

When one talks of inventory control, one is concerned with the answering of questions like: What is the optimum amount of inventory to carry? What is the Economical Order Quantity (EOQ) or Economic Lot Size (ELS) for an order from a supplier or from a production? What system of controlling inventory should be used i.e. what should be the technique of inventory control?

The optimum size of inventory depends upon the needs of the production department. Some parts, materials and even completed products have to be kept on hand in order to absorb discontinuities in production and to handle uncertainty. Finished stocks may as well as be held to provide better service for customers.

Concerning the question of the Economic Lot Size (or how much to buy) it can be stated that if we buy large quantities we have certain advantages such as (a) minimum ordering costs; (b) minimum handling and transportation costs; (c) reduction in purchasing costs through quantity discounts and the like ones. On the other hand there are inventory holding costs or stock holding costs which may mount up with the purchase of large quantities. The object of inventory control is to achieve a good balance between the cost of holding stocks and the benefits from stock holding. Keeping both these aspects in view, and effort should be made to make the size of the purchase

order so large that further economies are more than offset by the cost of holding stocks. In other words, a compromise is to be sought between too small and too large a purchase order. That size of the purchase order which minimises the total costs involved is termed as the Economical Order Quantity or Economic Lot Size which mainly depends upon two types of costs, viz., the preparation costs and the inventory carrying costs.

#### 6.5.1 EOQ with Price Breaks

In this, we consider the class of inventory in which cost is a variable factor. When items are purchased in bulk some discount price is usually offered by the supplier. Such discounts are referred to as quantity discounts or price breaks.

If the price discount is available then total cost per unit of system and items would be

$$TC = RK_1 + \frac{R}{q}C_s + \frac{1}{2}qK_1 \times I$$

where  $K_1$  is the cost of manufacturing or purchasing per unit and  $I$  denotes the holding cost per unit.

$$\frac{dTC}{dq} = 0 \Rightarrow q = \sqrt{2C_s R / K_1 I}$$

$$\frac{d^2TC}{dq^2} > 0 \text{ at } q = \sqrt{2C_s R / K_1 I}$$

∴ The optimum value of  $q$ , we have  $q^* = \sqrt{\frac{2C_s R}{K_1 I}}$

Now we proceed to consider an EOQ problem when the purchase cost is subject to price breaks.

#### Case I: EOQ Problems with One Price Break

The procedure for obtaining EOQ for a single discount is given as follows.

Order Quantity	Purchase Cost Per Unit
$0 \leq q_1 < b$	$K_{11}$
$B \leq q_2$	$K_{12}$

where  $b$  is the quantity at and beyond which the quantity discount applies and  $K_{12} < K_{11}$ .

• **Step 1:** Compute  $q^*$ , i.e., optimum order quantity for the lowest price (highest discount) and compare it with quantity  $b$ .

If  $q^* \geq b$ , then place the orders for quantities of size  $q^*$  and obtain the discount. Otherwise, go to the next step.

• **Step 2:** If  $q^* < b$ , we cannot place order at the reduced price  $K_{12}$ . Therefore, in order to maintain the optimum order quantity we need only to compare the total inventory cost for  $q = q^*$  (for price  $K_{11}$ ) with  $q = b$ .

The values of  $TC(q^*)$  and  $TC(b)$  may be determined as follows.

$$TC(q_1^*) = RK_{11} + \frac{R}{q_1^*} C_3 + \frac{1}{2} q_1^* \times K_{11} \times I$$

$$TC(b) = RK_{12} + \frac{R}{b} C_3 + \frac{1}{2} b \times K_{12} \times I$$

If  $TC(q_1^*) > TC(b)$  then  $q_1^* = b$ , otherwise  $q_1^* = q_1^*$ .

**Example 6.29:** Find the optimum order quantity for a product for which the price break are as follows:

Quantity	Unit cost (₹)
$0 \leq q_1 \leq 500$	10.00
$500 \leq q_2$	9.25

The monthly demand for the product is 200 units, the cost of storage is 2% of the unit cost and the cost of ordering is ₹ 350.00.

**Solution:**

Given:

$$C_3 = ₹ 350$$

$$R = 200 \text{ units per month}$$

$$I = 2/100 = 0.02$$

$$K_{11} = ₹ 10.00, K_{12} = ₹ 9.25.$$

• **Step 1:** The highest discount available is 9.25.

So we compute

$$q_1^* = \sqrt{\frac{2RC_3}{K_{12}I}}$$

$$= \sqrt{\frac{2 \times 350 \times 200}{9.25 \times 0.02}} = 870 \text{ units}$$

Now  $q_2^* = 870$  units and  $b = 500$ .

We have  $q_2^* > b$ , ∴ The optimum purchase quantity is given by  $q^* = q_2^* = 870$  units.

**Example 6.30:** Find the optimum order quantity for a product, the price breaks of which are as follows:

Quantity	Unit Cost (₹)
$0 \leq q_1 \leq 800$	₹. 1.00
$800 \leq q_2$	₹. 0.98

The yearly demand for the product is 1600 units per year, the cost of placing an order is ₹ 5, the cost of storage is 10% per year.

**Solution:**

Given:

$$R = 1600 \text{ units per year}$$

$$C_3 = ₹ 5 \text{ per order}$$

$$I_c = 10\% = ₹ 0.10.$$

$$K_{11} = ₹ 1.00, K_{12} = ₹ 0.98$$

• **Step 1:** The highest discount available is ₹. 0.98 =  $K_{12}$ . So we compute  $q_2$  at  $K_{12}$

$$q_2^* = \sqrt{\frac{2 \times R \times C_3}{K_{12} \times I}} = \frac{2 \times 1600 \times 5}{0.98 \times 0.1} = 404 \text{ units}$$

Now  $q_2^* = 404$  units,  $b = 800$  units. we get the case  $q_2^* < b$ .

• **Step 2:** Considering  $K_{11} = ₹. 1.00$ , we find the optimum order quantity.

$$q_1^* = \sqrt{\frac{2RC_3}{K_{11} \times I}}$$

$$= \sqrt{\frac{2 \times 5 \times 1600}{1 \times 0.1}} = 400 \text{ units}$$

Given:

$$R = 10,000 \text{ units per year}$$

$$C_3 = ₹ 5 \text{ per order}$$

$$I = ₹. 0.1, K_{11} = 100, K_{12} = ₹ 95$$

Since  $q_1^* = 400$  and  $b = 800$ , we get  $q_1^* < b$ . We compare the optimum cost of procuring the least quantity which will entitle us a price break.

$$TC(q_1^*) = 1600 \times 1 + \frac{1600}{400} \times 5 + \frac{1}{2} \times 400 \times 1 \times 0.10 = ₹ 1640$$

$$TC(b) = 1600 \times 0.98 + \frac{1600}{400} \times 5 + \frac{1}{2} \times 800 \times 0.98 \times 0.10$$

$$= ₹ 1627.20$$

$$TC(q_1^*) = TC(b)$$

∴ Optimum purchase quantity is given by  $q^* = b = 800$  units.

**Example 6.31:** The annual demand of a product is 10,000 units. Each unit costs ₹ 100 for orders placed in quantities below 200 units but for orders of 200 or above the price is ₹ 95. The annual inventory holding cost is 10% of the value of the item and the ordering cost is ₹ 500 per order. Find the economic lot size.

**Solution:**

Quantity	Unit Cost (₹)
$0 \leq q_1 \leq 200$	100
$q_2 \geq 200$	095

Given

$$R = 1000 \text{ units per year}$$

$$C_3 = ₹ 5 \text{ per order}$$

$$I = ₹ 0.1, K_{11} = ₹ 100, K_{12} = ₹ 95$$

• **Step 1:** Compute the optimal order quantity for the lowest price (highest discount).

$$q_3^* = \sqrt{\frac{2RC_3}{I \times K_{12}}} = \sqrt{\frac{2 \times 10000 \times 5}{0.1 \times 95}}$$

## NOTES

$$= 102.59 = 103 \text{ units}$$

Now,  $q_2^* = 103$ ,  $b = 200$  indicates that  $q_2^* < b$ .

• Step 2: Considering now  $K_{11} = ₹ 100$ , the optimum order quantity  $q_1^*$  is obtained as follows.

$$q_1^* = \sqrt{\frac{2 \times 10000 \times 5}{0.1 \times 100}} = 100 \text{ units}$$

Since  $q_1^* = 100$ ,  $b = 200$  indicates  $q_1^* < b$ .

We compare the optimum cost,

$$TC(q_1^*) = 1000 \times 100 + \frac{10000}{100} \times 5 + \frac{1}{2} \times 100 \times 100 \times 0.1 = ₹ 10,1000$$

$$TC(q_2^*) = 1000 \times 95 + \frac{10000}{200} \times 3 + \frac{1}{2} \times 200 \times 95 \times 0.1 = ₹ 96,100$$

Since  $TC(q_1^*) > TC(q_2^*)$ , optimum purchase quantity  $q^* = 200$  units.

**Case II: EOQ Problem with Two Price Breaks**

When there are two breaks the situation is illustrated as follows.

Order Quantity	Unit Price (₹)
$0 \leq q_1 \leq b_1$	$K_{11}$
$b_1 \leq q_2 \leq b_2$	$K_{12}$
$b_2 \leq q_3$	$K_{13}$

where  $b_1$  and  $b_2$  are the quantities which determine the price breaks.

The procedure for finding the optimum order quantity is given in the following steps

- Step 1: Compute  $q_3^*$  and compare it with  $b_2$ .
- Step 2: If  $q_3^* \geq b_2$ , the optimum order quantity is  $q_3$ . If  $q_3^* < b_2$ , go to the next step.
- Step 3: Compute  $q_2^*$ . Since  $q_3^* < b_2$ ,  $q_1^* < b_2$  because  $q_1^* < q_2^* < \dots < q_n^*$ .

Thus, either  $q_2^* < b_1$  or  $b_1 \leq q_2^* < b_2$ .

• Step 4: If  $q_2^* < b_1$  and  $b_1 \leq q_2^* < b_2$  then the same procedure as in the case of one price break is to be followed, i.e., compare  $TC(q_2^*)$  and  $TC(b)$  and then determine the optimum quantity.

• Step 5: If  $q_2^* < b_2$  and  $q_2^* < b_1$  then compute  $q_1^*$  which will now satisfy the inequality  $q_1^* < b_1$ . Compare  $TC(q_1^*)$  with  $TC(b_1)$  and  $TC(b_2)$  so as to get the optimum purchase quantity.

**Example 6.32:** Find the optimal order quantity for a product for which the price breaks are as follows:

Order Quantity	Unit Price (₹)
$0 \leq q_1 \leq 500$	10.00
$500 \leq q_2 \leq 750$	09.25
$750 \leq q_3$	08.75

The monthly demand for the product is 200 units, the cost storage is 2% of the unit cost and the cost of ordering is ₹ 350.

**Solution:** Given:  $C_3 = ₹ 350$ ,  $I = ₹ 0.02$ ,  $R = 200$  units per month

$$q_3^* = \sqrt{\frac{2RC_3}{K_{13} \times I}} = \sqrt{\frac{2 \times 200 \times 350}{(8.75)(0.02)}} = 894 \text{ units}$$

Since  $q_3^* = 894 > b_2 = 750$

The optimum order quantity is given by  $q = q_3^* = 894$  units.

**Example 6.33:** Find the optimum order quantity for a product for which the price breaks are as follows:

Order Quantity	Unit Price (₹)
$0 \leq q_1 \leq 100$	₹ 20 unit
$100 \leq q_2 \leq 200$	₹ 18 unit
$200 \leq q_3$	₹ 16 unit

The monthly demand for the product is 400 units. The storage cost is 20% of the unit cost of the product and the cost of ordering is ₹ 25.

**Solution:**

Given  $R = 400$  units  
 $I = ₹ 0.20$   
 $C_3 = 25.00$

$$\text{Calculate } q_3^* = \sqrt{\frac{2RC_3}{K_{13} \times I}} = \sqrt{\frac{2 \times 400 \times 25}{16 \times 0.2}} = 79 \text{ units}$$

$q_3^* = 79 < b_2 = 200$ . Next we compute  $q_2$ .

$$q_2^* = \sqrt{\frac{2RC_3}{K_{12} \times I}} = \sqrt{\frac{2 \times 400 \times 25}{18 \times 0.2}} = 75 \text{ units.}$$

Again, since  $q_2^* < b_2$  and  $q_2^* < b_1 = 100$ , we calculate  $q_1^*$ .

$$q_1^* = \sqrt{\frac{2RC_3}{K_{11} \times I}} = \sqrt{\frac{2 \times 400 \times 25}{20 \times 0.2}} = 70 \text{ units}$$

Compare  $TC(q_1^*)$  with  $TC(b_1)$  and  $TC(b_2)$  in order to find the optimum order quantity.

$$TC(q_1) = 400 \times 20 + \frac{400}{70} \times 25 + \frac{1}{2} \times 70 \times 20 \times 0.2 = ₹ 8282.86$$

$$TC(b_1) = 400 \times 18 + \frac{400}{100} \times 25 + \frac{1}{2} \times 100 \times 18 \times 0.2 = ₹ 7480.00$$

$$TC(q_2) = 400 \times 16 + \frac{400}{200} \times 25 + \frac{1}{2} \times 200 \times 16 \times 0.2 = ₹ 6770.00$$

Since  $TC(b_1) > TC(q_2)$ , the optimum order quantity is given by  $q^* = b_2 = 200$  units.

Example 6.34: Find the optimal order quantity for a product for which the price breaks are as follows

Order Quantity	Unit Price (₹)
$0 \leq q_1 \leq 500$	100
$500 \leq q_2 \leq 750$	92.50
$750 \leq q_3$	87.50

The monthly demand of the product is 200 units, the holding cost is 2% of the unit cost and the ordering cost is ₹ 1000.

Solution:

Given,  $R = 200$  units per month  
 $I = 0.02$   
 $C_3 = ₹ 1000$   
 $K_{11} = ₹ 100, K_{12} = ₹ 92.50, K_{13} = 87.50$

We calculate  $q_3^* = \sqrt{\frac{2RC_3}{K_{13} \times I}}$   
 $= \sqrt{\frac{2 \times 200 \times 1000}{87.5 \times 0.02}} = 478$  units

Since  $q_3^* = 478 < b_2 = 750$ , we next calculate

$$q_2^* = \sqrt{\frac{2RC_3}{K_{12} \times I}} = \sqrt{\frac{2 \times 200 \times 1000}{92.5 \times 0.02}} = 465$$
 units

Also, since  $q_2^* < b_2$  and  $q_2^* < b_1 = 500$

We calculate  $q_1^* = \sqrt{\frac{2RC_3}{K_{11} \times I}} = \sqrt{\frac{2 \times 200 \times 1000}{100 \times 0.02}} = 447$

Next, we compare  $T_c(q_1^*)$  with  $T_c(b_1)$  and  $T_c(b_2)$  to get the optimum order quantity.

$$TC(q_1^*) = RK_{11} + \frac{R}{q_1^*} C_3 + \frac{1}{2} q_1^* \times K_{11} \times I$$

$$= 200 \times 100 + \frac{200}{447} \times 1000 + \frac{1}{2} \times 447 \times 100 \times 0.02$$

$$= ₹ 20894$$

$$TB(b_1) = RK_{12} + \frac{R}{b_1} C_3 + \frac{1}{2} b_1 \times K_{12} \times I$$

$$= 200 \times 92.5 + \frac{200}{500} \times 1000 + \frac{1}{2} \times 500 \times 92.5 \times 0.02$$

$$= ₹ 19362.50$$

NOTE

$$TB(b_2) = RK_{13} + \frac{R}{b_2} C_3 + \frac{1}{2} b_2 \times K_{13} \times I$$

$$= 200 \times 87.5 + \frac{200}{750} \times 1000 + \frac{1}{2} \times 750 \times 87.5 \times 0.02$$

$$= ₹ 18422.90$$

Since,  $Tc(q_1^*) > Tc(b_1) > Tc(b_2)$  the optimal order quantity is  $q = b_2 = 750$  units. The optimal order quantity corresponds to the lowest total inventory cost.

Example 6.35: A shopkeeper has a uniform demand of an item at the rate of 50 item per month. He buys from a supplier at a cost of ₹ 6 per item and the cost of ordering is ₹ 10 for each order. If the stock holding costs are 20 per cent per year of stock value, how frequently should he replenish his stocks.

Now, suppose the supplier offers a 5 per cent discount on orders between 200 and 999 items and a 10 per cent discount on orders exceeding or equal to 1000 items. Can the shopkeeper reduce his costs by taking advantage of either of these discounts?

Solution:

Given,  $R = 100$  items per year  
 $C_3 = ₹ 10$  per order  
 $C_1 = ₹ 1.20$  (i.e.,  $6 \times 0.2$ )

$$q^* = \sqrt{\frac{2C_3 R}{C_1}} = \sqrt{\frac{2 \times 100 \times 10}{1.2}} = 100$$
 items

$$t^* = \frac{q^*}{R} = \frac{100}{600} = \frac{1}{6}$$
 year = 2 months

The total annual cost includes the fixed cost, the set-up cost and the holding costs.

Fixed cost = ₹ 3600 (i.e.,  $6 \times 600$ )

100 items are ordered each time. This means that a total of 6 orders are placed in a year. Hence, the replenishment cost is ₹ 60.00.

Average inventory cost throughout the year

∴ Average inventory holding cost =  $50 \times 0.2 \times 6 = ₹ 60.00$   
Hence, the total cost = ₹ 3600 + 60 + 60 = ₹ 3720.00.

In the case of discounts we have the following formulation:

Quantity	Unit Cost (₹)
$0 \leq q_1 < 200$	6.00
$200 \leq q_2 < 1000$	5.70 (5% discount)
$1000 \leq q_3$	5.40 (10% discount)

$$q_3^* = \sqrt{\frac{2C_3 R}{K_{13} I}} = \sqrt{\frac{2 \times 100 \times 600}{5.4 \times 0.20}} = 105$$
 units

$q_3^* < b_2 = 1000$ , we next compute  $q_2^*$ .

$$q_2^* = \sqrt{\frac{2C_3 R}{K_{12} I}} = \sqrt{\frac{2 \times 100 \times 600}{5.7 \times 0.2}} = 103$$
 units

Again, since  $q_2^* < b_1$ , we compute

$$q_1^* = \sqrt{\frac{2C_3R}{K_{11}I}} = \sqrt{\frac{2 \times 10 \times 600}{6 \times 0.2}} = 100 \text{ units}$$

To find the optimum order quantity, we compare  $Tc(q_1)$  with  $Tc(b_1)$  and  $Tc(b_2)$ .

$$Tc(q_1) = 10 \times \frac{600}{100} + 600 \times 6 + 0.2 \times 6 \times \frac{100}{2} = ₹ 3720$$

$$Tc(b_1) = 10 \times \frac{600}{200} + 600 \times 5.7 + 0.2 \times \frac{5.7 \times 200}{2} = ₹ 3564$$

$$Tc(b_2) = 10 \times \frac{600}{1000} + 600 \times 5.4 + \frac{0.2 \times 5.4 \times 1000}{2} = ₹ 3786$$

$$\therefore Tc(b_1) < Tc(q_1) < Tc(b_2).$$

Hence, the optimum purchase quantity is  $q^* = b_1 = 200$  units. The shopkeeper should accept the offer at 5 per cent discount only as by doing this he is able to save ₹ 3720 - 3564 = ₹ 156 during the year.

#### EOQ Problem with $n$ Price Breaks

When there are  $n$  price breaks the situation may be illustrated as follows:

Quantity	Unit Cost (₹)
$0 \leq q_1 < b_1$	$K_{11}$
$b_1 \leq q_2 < b_2$	$K_{12}$
$\vdots$	$\vdots$
$b_{n-1} \leq q_n$	$K_{1n}$

where  $b_1, b_2, b_{n-1}$  are those quantities which determine the price breaks.

The procedure for obtaining the optimum order quantity in this case will be as follows.

• **Step 1:** Compute  $q_n^*$ . If  $q_n^* \geq b_{n-1}$ , the optimum order quantity is reached, i.e.,  $q_n^*$

• **Step 2:** If  $q_n^* < b_{n-1}$ , compute  $b_{n-1}^{**} = b_{n-2}$ , proceed as in the case of one price break. The optimum order quantity is determined by comparing  $Tc(b_{n-1}^*)$  with  $Tc(b_{n-2})$ .

• **Step 3:** If  $q_n^* < b_{n-2}$ , compute  $b_{n-2}^{**} = b_{n-3}$ , proceed as in the case of two price breaks, i.e., the optimum order quantity is determined by comparing  $Tc(b_{n-2}^*)$  with  $Tc(b_{n-3})$  and  $Tc(b_{n-2})$ .

• **Step 4:** If  $q_n^* < b_{n-3}$ , compute  $q_{n-3}^* \geq b_{n-4}$ . Then compare  $Tc(b_{n-3}^*)$  with  $Tc(b_{n-4})$ ,  $Tc(b_{n-2})$  and  $Tc(b_{n-1})$ .

• **Step 5:** Compute in this way unit  $q_{n-j}^* \geq (b_{n-j} + 1)$   $0 \leq j \leq n-1$  then compare  $Tc(q_{n-j}^*)$  with  $Tc(q_{n-j-2})$  until  $q_{n-j}^* \geq q_{n-j+1}^* Tc(q_{n-1})$ .

**Example 6.36:** Find the optimum order quantity for a product for which the price breaks are as follows:

Quantity	Unit Cost (₹)
0-499	25
500-1499	24.80
1500-2999	24.60
Over 3000	24.40

Ordering cost is ₹ 180 per order, carrying cost is 0.10. Demand is 500 units per year. Also, find the minimum inventory cost.

**Solution:**

Given,  $D = 500$  units/year

$C_3 = ₹ 180$ /order

$I = 0.10$ .

$K_{11} = 25, K_{12} = 24.80, K_{13} = 24.60, K_{14} = 24.40$

$b_1 = 500, b_2 = 1500, b_3 = 3000$ .

$$q_4^* = \sqrt{\frac{2RC_3}{K_{14} \times I}} = \sqrt{\frac{2 \times 500 \times 180}{0.1 \times 24.4}} = 272 \text{ units}$$

As  $q_4^* = 272 < b_3 = 3000$ , next we compute  $q_3$ .

$$q_3^* = \sqrt{\frac{2RC_3}{K_{13} \times I}} = \sqrt{\frac{2 \times 500 \times 180}{0.10 \times 24.60}} = 270 \text{ units}$$

$\therefore q_3^* = 270 < b_2 = 1500$ , next we

$$\text{Calculate } q_2^* = \sqrt{\frac{2RC_3}{K_{12} \times I}} = \sqrt{\frac{2 \times 500 \times 180}{0.1 \times 25}} = 269 \text{ units}$$

$$\text{Next, we compute } q_1^* = \sqrt{\frac{2RC_3}{K_{11} \times I}} = \sqrt{\frac{2 \times 500 \times 180}{0.1 \times 24.8}} = 268 \text{ units}$$

$$\text{Now, } Tc(q_1^*) = RK_{11} + \frac{R}{q_1^*} \times C_3 \frac{q_1^*}{2} (I \times K_{11})$$

$$= ₹ 13170.82$$

$$Tc(b_1) = RK_{12} + \frac{R}{b_1} \times C_3 \frac{b_1}{2} (I \times K_{12})$$

$$= 13200.00$$

$$Tc(b_2) = RK_{13} + \frac{R}{b_2} \times C_3 \frac{b_2}{2} (I \times K_{13})$$

$$= ₹ 14205.00$$

$\therefore Tc(q_1^*) > Tc(b_1) > Tc(b_2)$ ,  $q_1^* = 268$  units

This is the optimum order quantity.

Minimum inventory cost = ₹ 13170.82

## 6.6 SUMMARY

- Number of customers waiting for service form a queue and it does not include persons or customers being served.
- A queueing system can be completely described by the input (arrival pattern), the service mechanism (service pattern), the queue discipline and customer's behaviour.
- Input describes the way in which the customers arrive and join the system. Generally, customers arrive in a more or less random fashion, which is not

## NOTES

NOTES

possible to predict. Thus, the arrival pattern can be described in terms of probabilities, and consequently, the probability distribution for inter-arrival times (the time between two successive arrivals) must be defined.

- Anything that arrives for service is known as arrival unit; arrival unit may be a person, a machine, a vehicle, etc., which comes for taking some kind of service.
- Customers arrive in random described in terms of probability where arrival rate follows a Poisson distribution and mean arrival rate is taken in a specified time interval and denoted by  $\lambda$ .
- Service time for attending a customer follows a negative exponential distribution and the mean service rate is given by number of customers serviced in a specified interval of time and is denoted as  $\mu$ .
- If there are an infinite number of servers, then all the customers are served instantaneously on arrival and there will be no queue.
- If the number of servers is finite then the customers are served according to a specific order, with service time as a constant or random variable.
- Distribution of service time that is important in practice is the negative exponential distribution. The mean service rate is denoted by  $m$ .
- The customers are selected according to specific rules for service when a queue has been formed. The most common disciplines are First Come First Served (FCFS), First In First Out (FIFO), Last In First Out (LIFO) and selection for Service In Random Order (SIRO).
- Under priority discipline, the service is of two types, namely pre-emptive and non-pre-emptive. In pre-emptive system, the high priority customers are given service over the low priority customers; in non-pre-emptive system, a customer of low priority is serviced before a customer of high priority. In the case of parallel channels 'fastest server rule' is adopted.
- In balking, a customer who leaves the queue because the queue is too long and he has no time to wait or does not have sufficient waiting space.
- Reneging occurs when a waiting customer leaves the queue due to impatience.
- In certain applications some customers are served before others, regardless of their arrival. These customers have priority over others.
- Customers may jockey from one waiting line to another. This is most common in a supermarket.
- A system is said to be in a transient state when its operating characteristics are dependent on time.
- A steady state system is the one in which the behaviour of the system is independent of time.
- An important measure of a simple queue is its traffic intensity given by,

$$\text{Traffic Intensity } r = \frac{\text{Mean arrival rate}}{\text{Mean service rate}} = \frac{\lambda}{\mu}$$

Erlang.

- Generally, queuing model may be completely specified in the symbol form  $(a/b/c): (d/e)$  where,
  - $a$  = Probability law for the arrival (inter-arrival) time.
  - $b$  = Probability law according to which the customers are being served.
  - $c$  = Number of channels (or service stations).
  - $d$  = Capacity of the system, i.e., the maximum number allowed in the system (in service and waiting).
  - $e$  = Queue discipline.
- In FCFS or FIFO queuing discipline, a person is given service according to their arrival, i.e., those who arrive first are served first.
- In LIFO or LCFS queuing discipline in which services are provided in the order that is reverse of FIFO. A stack is an example of LIFO.
- In SIRO queuing discipline, service is provided in random order, irrespective of arrival of customers
- Model I (M/M/1) or ( $\infty$ /FCFS) denotes Poisson arrival (exponential inter-arrival), Poisson departure (exponential service time), Single server, Infinite capacity and First come first served service discipline. The letter  $M$  is used due to Markovian property of exponential process.
- Model II (M/M/1): (N/FCFS) Model II (M/M/1): (N/FCFS) model differs from model I in the sense that, the maximum number of customers in the system is limited to  $N$ . Therefore, the difference equations of model I are valid for this model as long as  $n < N$ . Arrivals will not exceed  $N$  in any case.
- Model III: Multiservice Model (M/M/S): ( $\infty$ /FCFS) model takes the number of service channels as  $S$ .
- Model IV (M/M/S): (M/FCFS) Model IV (M/M/S): (M/FCFS) is essentially the same as model III, except that the maximum number of customers in the system is limited to  $N$ , where  $N > S$  ( $S$  = Number of channels).
- The random variable is a real-valued function, defined over a sample space associated with the outcome of a conceptual chance experiment. Random variables are classified according to their probability density function.
- Random numbers are called pseudo-random numbers when they are generated by some deterministic process, but have already qualified the pre-determined statistical test for randomness.
- The Monte Carlo method is a simulation technique in which statistical distribution functions are created by using a series of random numbers. The method is generally used to solve the problems that cannot be adequately represented by the mathematical models or where the solution of the model cannot be arrived at, by analytical method.
- Monte Carlo simulation needs the generation of a sequence of random numbers, which constitute an integral part of the simulation model and also help in determining random observations from the probability distribution.

Self-Ins...

- Simulation is the process of designing a mathematical or logical model of a real system and then accomplishing experiments with the model to describe, explain and predict the behavior of the real system.
- Generally, the systems are referred to be dynamic as their status changes over time. Hence, to explain this status the concept of the state of a system is used.
- A static simulation model represents a system at a particular point in time whereas a dynamic simulation model represents a system developed over time.
- Based on these two classifications, a simulation may be classified as deterministic or stochastic. A deterministic simulation model is one that contains no random variables whereas a stochastic simulation model contains one or more random variables.
- When the state of system transforms/changes only at discrete points in time termed as 'discrete events'. When the state of system changes continuously over time, then it is termed as 'continuous event'.
- Time is advanced from event to event. The future events list illustrates the ordered list of upcoming events. The moment events are scheduled, they are added to the event list. Similarly, when events occur they are removed from event list.
- In simulation process the inverse transformation method can be described on the basis of two postulates: inverse transform method for simulating continuous random variables and inverse transform method for simulating discrete random variables.
- By using a fresh series of random numbers at the appropriate junctures we can also examine the reactions of the simulated model just as if the same alterations had actually been made in the system itself. Monte Carlo simulation, therefore, provides a tool of knowing in advance whether or not the expense to be incurred or the investment to be made in making the changes envisaged.
- The main purpose of simulation in management is to provide feedback, which is vital for the learning process. It creates an atmosphere in which managers play a dynamic role by enriching their experience through involvement in reckoning with actual conditions through experimentation on paper.
- Simulation is a representation of reality through the use of a model or other device which will react in the same manner as reality under a given set of conditions.
- Simulation is also defined as the use of a system model that has the designed characteristics of reality in order to produce the essence of actual operation.
- Monte Carlo simulation has been applied to a wide diversity of problems ranging from queuing process, inventory problem, risk analysis concerning a major capital investment such as the introduction of a new product, expansion of the capacity, and many other problems.

- Monte Carlo methods are basically the algorithms used in the computation of result to be calculated from repeated random sampling. These methods help in computerized calculations because these can perform repeated computation using random or pseudo-random numbers.
- 'Monte Carlo' is the code name given by John von Neumann and S.M. Ulam to the technique of solving problems though it is too expensive for experimental solutions and too complicated for analytical treatment.
- Monte Carlo simulation uses random number tables to reproduce on paper the operation of any given system under its own working conditions. This technique is used to solve problems that depend upon probability where formulation of mathematical model is not possible.
- Inventory is defined as any idle resources of an enterprise. It is a physical stock of goods kept for future use. In a factory, the inventory may be in the form of raw materials, parts, semi-finished goods. Inventory also includes furniture, machinery, etc.
- The need of the management to make decisions regarding the inventory arises because of the various alternative courses of action available with the enterprise.
- Item cost refers to the cost associated with an item whether it is manufactured or purchased. The purchase price will be considered when discounts are allowed for any purchase above a certain quantity.
- The penalty costs that are incurred as a result of running out of stock (i.e., shortage) are known as shortage or stock-out costs. These are denoted by  $C_2$  per unit of goods for a specified period.
- Demand refers to the number of items required per period. It may be known exactly or known in terms of probabilities or may be completely unknown.
- The demand pattern of items may be either deterministic or probabilistic. Problems in which demand is known and fixed are called deterministic problems. Whereas those problems in which the demand is assumed to be a random variable are called stochastic or probabilistic problems.
- In case of deterministic demand, it is assumed that the quantities needed over subsequent periods of time are known exactly. Further, the known demand may be fixed or variable with time. Such demands called state or dynamic demands respectively.
- The probabilistic demand occurs when the demand over a certain period of time is not known with certainty, but it is described by a known probability distribution. A probabilistic demand may be either stationary or non-stationary over time.
- The time gap between the placing of an order and the actual arrival of the inventory is known as lead time. If the lead time is known and is not equal to zero, and if the demand is deterministic, all that one requires to do is to order in advance by the time equal to the lead time. If the lead time is zero, there is no need to order in advance.

- The optimum size of inventory depends upon the needs of the production department.
- The object of inventory control is to achieve a good balance between the cost of holding stocks and the benefits from stock holding. Keeping both these aspects in view, and effort should be made to make the size of the purchase order so large that further economies are more than offset by the cost of holding stocks.

## 6.7 KEY TERMS

- **Arrival unit:** Anything that arrives for service is known as arrival unit; arrival unit may be a person, a machine, a vehicle, etc., which comes for taking some kind of service
- **Queue:** Number of customers waiting for service form a queue and it does not include persons or customers being served
- **Mean arrival rate:** Customers arrive in random described in terms of probability where arrival rate follows a Poisson distribution and mean arrival rate is taken in a specified time interval and denoted by  $\lambda$
- **Mean service rate:** Service time for attending a customer follows a negative exponential distribution and the mean service rate is given by number of customers serviced in a specified interval of time and is denoted as  $\mu$
- **Traffic intensity:** The ratio of mean arrival rate to mean service rate and denoted by  $\rho = \lambda/\mu$
- **Service pattern:** Arrangement of service facilities to serve customers
- **Queue discipline:** A rule according to which customers are selected when a queue is formed; the most common disciplines are FCFS (First Come First Served), LIFO (Last In First Out), SIRO (Selection of Service In Random Order) and Priority Queue
- **Priority queue:** In this queueing discipline, a customer with high priority is served first irrespective of order of arrival. But customers having same priorities are served according to FCFS
- **Balking:** A type of customer behaviour where the customer leaves the queue finding it very long and has no time to wait
- **Reneging:** A type of behaviour shown by customer when he stands in the queue but leaves after sometime due to impatience
- **Jockeying:** A type of behaviour shown by a customer when he moves from one queue to another, expecting quicker service
- **Transient and steady states:** A system is said to be in transient state when its operating characteristics are dependent on time. It is called steady state when such operating characteristics are independent of time
- **FCFS or FIFO:** In this queueing discipline, a person is given service according to their arrival, i.e., those who arrive first are served first
- **LIFO or LCFS:** This is a queueing discipline in which services are provided in the order that is reverse of FIFO, a stack is an example of LIFO

- **SIRO:** In this queueing discipline, service is provided in random order, irrespective of arrival of customers
- **Model I [(M/M/1): ( $\infty$ /FCFS)]:** This model denotes arrival in exponential time interval, departure (after getting the service) in exponential service time, single server, infinite capacity and FCFS shows queueing discipline
- **Model II [(M/M/1): (N/FCFS)]:** This model denotes arrival in exponential time interval, departure (after getting the service) in exponential service time, single server, finite capacity and FCFS shows queueing discipline
- **Model III [(M/M/S): ( $\infty$ /FCFS)]:** This model denotes arrival in exponential time interval, departure (after getting the service) in exponential service time, having S number of servers, infinite capacity and FCFS shows queueing discipline
- **Model IV [(M/M/S): (N/FCFS)]:** This model denotes arrival in exponential time interval, departure (after getting the service) in exponential service time, having S number of servers, finite capacity and FCFS shows queueing discipline
- **Simulation:** A representation of reality using a model or other device that reacts in the same manner as reality under a given set of conditions and is an imitation of a reality
- **Random number:** It refers to a number assigned to a random variable following uniform probability density function
- **System:** A collection of entities that act and interact toward the accomplishment of some logical end
- **Event:** A situation that causes the state of the system to change instantaneously
- **Event list:** A list in which all the information regarding change of state is preserved
- **Pseudorandom numbers:** These random numbers are generated by some deterministic process but have already qualified the predetermined statistical test for randomness
- **Monte carlo technique:** A technique of simulation in which a series of random numbers are used to create statistical distribution function and used where mathematical representation of a problem is not found adequate

## 6.8 ANSWERS TO 'CHECK YOUR PROGRESS'

1. Number of customers waiting for service form a queue. This does not include persons or customer being served.
2. In this queueing discipline, a person is given service according to their arrival. Those who arrive first are served first. FCFS or First Come First Served.
3. This is a queueing discipline in which service is provided in random order, irrespective of arrival of customers.

4. Essential components of a queuing discipline are:
  - Arrival pattern
  - Service pattern
  - Queuing discipline
  - Customer behaviour
5. Balking is a type of customer behaviour when he leaves the queue finding queue very long and has no time to wait.
6. Reneging is a type of behaviour shown by customer when he stands in the queue but leaves after sometime due to impatience.
7. Jockeying type of behaviour is shown by customer when he moves from one queue to another, expecting quicker service.
8. A system is in transient state when its operating characteristics depend on time. It is called steady state when such operating characteristics are independent of time.
9. Customers arrive in random way and is described in terms of probability. Arrival rate follows a Poisson distribution and mean arrival rate is taken in a specified time interval and denoted by  $\lambda$ .
10. Service time for attending a customer follows a negative exponential distribution. Mean service rate is given by number of customers serviced in a specified interval of time and is denoted as  $\mu$ .
11. Traffic intensity is given by the ratio of mean arrival rate to mean service rate and denoted by  $\rho = \lambda/\mu$ .
12. Queuing models are described in symbolic form as  $(a/b/c): (d/e)$ , where
  - $a$  stands for probability law for arrival.
  - $b$  stands for law according to which customers are served.
  - $c$  stands for number of service channels.
  - $d$  stands for capacity of the system (maximum number allowed in the system).
  - $e$  stands for queuing discipline.
13. The model  $(M/M/1): (N/F\text{ CFS})$  denotes arrival in exponential time interval, departure (after getting the service) in exponential service time, single server with finite capacity and FCFS shows queuing discipline.
14.  $(M/M/1): (N/FCFS)$  and  $(M/M/S): (\infty/FCFS)$  are two models that has FCFS as queuing discipline. If first model arrival is in exponential time interval, departure (after getting the service) in exponential service time, single server, finite capacity and FCFS shows queuing discipline. In the second model, arrival is in exponential time interval, departure (after getting the service) in exponential service time, having  $S$  number of servers, infinite capacity and FCFS shows queuing discipline.
15. Random variables are classified according to their probability density function.

16. Random numbers are called pseudo-random numbers when they are generated by some deterministic process, but have already qualified the pre-determined statistical test for randomness.
17. Monte Carlo simulation needs the generation of a sequence of random numbers, which constitute an integral part of the simulation model and also help in determining random observations from the probability distribution.
18. Simulation technique is specifically used to imitate the function of a real world system that develops over time.
19. A system is a collection of entities that act and interact toward the accomplishment of some logical end. Generally, the systems are referred to be dynamic as their status changes over time.
20. When the state of system transforms/changes only at discrete points in time termed as 'discrete events'. When the state of system changes continuously over time, then it is termed as 'continuous event'.
21. A Linear Congruential Generator (LCG) represents one of the oldest and well known pseudorandom number generator algorithms.
22. The main purpose of simulation in management is to provide feedback, which is vital for the learning process.
23. Monte Carlo methods are basically the algorithms used in the computation of result to be calculated from repeated random sampling. These methods help in computerized calculations because these can perform repeated computation using random or pseudo-random numbers.
24. Analytical solution produces the optimal answer to a given problem, while Monte Carlo simulation yields a solution which should be very close to the optimal but not necessarily the exact correct solution.
25. Inventory is defined as any idle resources of an enterprise. It is a physical stock of goods kept for future use. In a factory, the inventory may be in the form of raw materials, parts, semi-finished goods. Inventory also includes furniture, machinery, etc.
26. There are four categories of inventory cost associated with keeping inventories of items. They are as follows:
  - (i) Item (or production or purchase) cost
  - (ii) Ordering or set-up cost
  - (iii) Carrying or holding cost
  - (iv) Shortage or stock out cost

## 6.9 QUESTIONS AND EXERCISES

### Short-Answer Questions

1. How queue is formed?
2. What are arrival rates?

3. What is mean service rate?
4. What is traffic intensity?
5. How a queuing model is denoted using symbols?
6. What is meant by (M:M:S): (N/FCFS)?
7. When simulation is used? What is done in a simulation?
8. Differentiate between a random number and a pseudo-random number.
9. When can we resort to the Monte Carlo technique?
10. Differentiate between deterministic and stochastic simulation model.
11. How simulation is initiated?
12. What is fixed-increment time-advance method?
13. Who gave the code name Monte Carlo?
14. What can you do using the Monte Carlo method?
15. Can simulation be applied to every situation? When can it not be applied?
16. Do you agree that simulation is a modern day need? Give reason for your answer.
17. What are the reasons for maintaining inventories?
18. What are the types of inventories?

#### Long-Answer Questions

1. People arrive at a theatre ticket centre in a Poisson distributed arrival rate of 25 per hour. Service time is constant at two minutes. Calculate,
  - (i) The mean number in the waiting line.
  - (ii) The mean waiting time.
  - (iii) Utilization factor.
2. At a one-man barber shop, customers arrive according to Poisson distribution, with a mean arrival rate of five per hour and the hair cutting time was exponentially distributed with an average hair cut taking 10 minutes. It is assumed that because of his excellent reputation, customers were always willing to wait. Calculate the following:
  - (i) Average number of customers in the shop and the average number waiting for a hair cut.
  - (ii) The per cent of time, an arrival can walk right in without having to wait.
  - (iii) The percentage of customers who have to wait before getting into the barber's chair.
3. Cars arrive at a petrol pump with exponential inter-arrival time having mean  $\frac{1}{2}$  minute. The attendant on an average takes  $\frac{1}{3}$  minute per car to supply petrol. The service time being exponentially distributed, determine,
  - (i) The average number of cars waiting to be served.
  - (ii) The average number of cars in the queue.
  - (iii) The proportion of time, for which the pump attendant is idle.

4. The mean arrival rate to a service centre is three per hour. The mean service time is found to be 10 minutes per service. Assuming Poisson arrival and exponential service time, find,
  - (i) The utilization factor for this service facility.
  - (ii) The probability of two units in the system.
  - (iii) The expected number of units in the system.
  - (iv) The expected time in minutes that a customer has to spend in the system.
5. At a public telephone booth in a post office, arrivals are considered to be Poisson, with an average inter-arrival time of 12 minutes. The length of the phone call may be assumed to be distributed exponentially with an average of four minutes. Calculate the following.
  - (i) What is the probability that a fresh arrival will not have to wait for the phone?
  - (ii) What is the probability that an arrival will have to wait more than 10 minutes before the phone is free?
  - (iii) What is the average length of the queue that forms from time to time?
6. Consider a self-service store with one cashier. Assume Poisson arrivals and exponential service time. Assuming nine customers arrive on the average of every five minutes and the cashier can serve 10 of them in five minutes, find,
  - (i) (a) The average number of customers queuing for service.
  - (b) The average time a customer spends in the system.
  - (c) The average time a customer waits before being served.
  - (ii) The probability of having more than 10 customers in the system.
  - (iii) The probability that a customer has to queue for more than 20 minutes. If the service can be increased up to 12 in every five minutes by using a different cash register, what will be the effect on the quantities (a), (b) and (c).
7. Patients arrive at a clinic according to a Poisson distribution at the rate of 30 patients per hour. The waiting room does not accommodate more than 14 patients. Examination time per patient is exponential with mean rate 20 per hour.
  - (i) Find the effective arrival rate at the clinic.
  - (ii) What is the probability that an arriving patient will not wait? Will he find a vacant seat in the room?
  - (iii) What is the expected waiting time until a patient is discharged from the clinic?
8. A stenographer has 5 persons for whom she performs stenographic work. Arrival rate is Poisson and service times are exponential. Average arrival rate is 4 per hour with an average service time of 10 minutes. Find,
  - (i) The average waiting time of an arrival.
  - (ii) The average length of waiting line.
  - (iii) The average time on arrival spent in the system.

9. Customers arrive at a one-window drive-in bank according to Poisson distribution with mean 10 per hour. Service time per customer is exponential with mean 5 minutes. The car space in front of the window, including that for the service, can accommodate a maximum of 3 cars. Other cars can wait outside this space.
- What is the probability that an arriving customer can drive directly to the space in front of the window?
  - What is the probability that an arriving customer will have to wait outside the indicated space?
  - How long is an arriving customer expected to wait before starting the service?
10. A two-channel waiting line with Poisson arrivals has a mean arrival rate of 50 per hour and exponential service with a mean service rate of 75 per hour for each channel. Find,
- The probability of an empty system.
  - The probability that an arrival in the system will have to wait.
11. There are two clerks in a university, to receive dues from the students. If the service time for each student is exponential with mean four minutes and if the boys arrive in a Poisson fashion at the counter at the rate 10 per hour, determine,
- The probability of having to wait for service.
  - The expected percentage of idle time for each clerk.
12. A general insurance company has three claim adjusters in its branch office. People with claims against the company are found to arrive in a Poisson fashion at an average rate of 20 per eight-hour day. The amount of time that an adjuster spends with a claimant is found to have a negative exponential distribution with mean service time 40 minutes. Claimants are processed in the order of appearance.
- How many hours a week can an adjuster expect to spend with the claimants?
  - How much time on an average does a claimant spend in the branch office?
13. A railway goods traffic station has four claims assistants. Customers with claims against the railway are observed to arrive in a Poisson fashion at an average rate of 24 per eight-hour day for six days. The amount of time the claims assistant spends with the claimant is found to have an exponential distribution with a mean service time of 40 minute. Service is given in the order of appearance of the customers.
- How many hours/week can a claim assistant expect to serve the claimant?
  - On an average, how much time does a claimant spend in the goods traffic office?
14. Explain simulation and its various types.
15. What is a random variable? Why is it used? Explain with the help of an example.
16. Explain the methodology of the Monte Carlo simulation techniques.

17. Describe the role of simulation in management process.
18. Discuss Monte Carlo simulation with reference to queuing theory, inventory control and production line.
19. The following data is observed in a tea serving counter. The arrival is for one minute interval.

No. of Persons Arriving	0	1	2	3	4	5
Probability	0.05	0.15	0.40	0.20	0.15	0.05

The service is taken as 2 person for one minute interval. Using the following random numbers simulate for 15 minutes period.

09, 54, 94, 01, 80, 73, 20, 26, 90, 79, 25, 48, 99, 25, 89

Calculate also the average number of persons waiting in the queue per minute.

20. A special purpose drill bores holes having a mean diameter of 1 cm. The process is normally distributed. Simulate a sequence of 10 diameters if the standard deviation of the process is 0.002 cm.
21. At a toll office a sample of 100 arrivals of vehicles gives the following frequency distribution of the inter arrival and service time (calculate as minutes).

Inter Arrival Time	Frequency	Service Time	Frequency
1.0	2	1.5	10
1.5	5	-	-
2.0	9	2	22
2.5	25	-	-
3.0	22	2.5	40
3.5	11	-	-
4.0	10	3.0	20
4.5	6	-	-
5.0	3	3.5	8
5.5	2	-	-

There is a clerk at the office. Simulate the process for 20 arrivals and estimate the average percentage vehicle waiting time and average per cent idle time of the clerk.

### 6.10 FURTHER READING

Taha, H.A. 2006. *Operations Research: An Introduction*, 8th edition. New Delhi: Prentice-Hall of India.

Jensen, Paul A., and Jonathan F. Bard. 2003. *Operations Research Models and Methods*. New York: John Wiley & Sons.

Gupta, P.K., and D.S. Hira. 2002. *Introduction to Operations Research*. New Delhi: S. Chand & Co.

Kothari, C.R. 1982. *An Introduction to Operational Research*. New Delhi: Vikas Publishing House Pvt. Ltd.

Kalavathy, S. 2013. *Operations Research*, 4th edition. New Delhi: Vikas Publishing House Pvt. Ltd.